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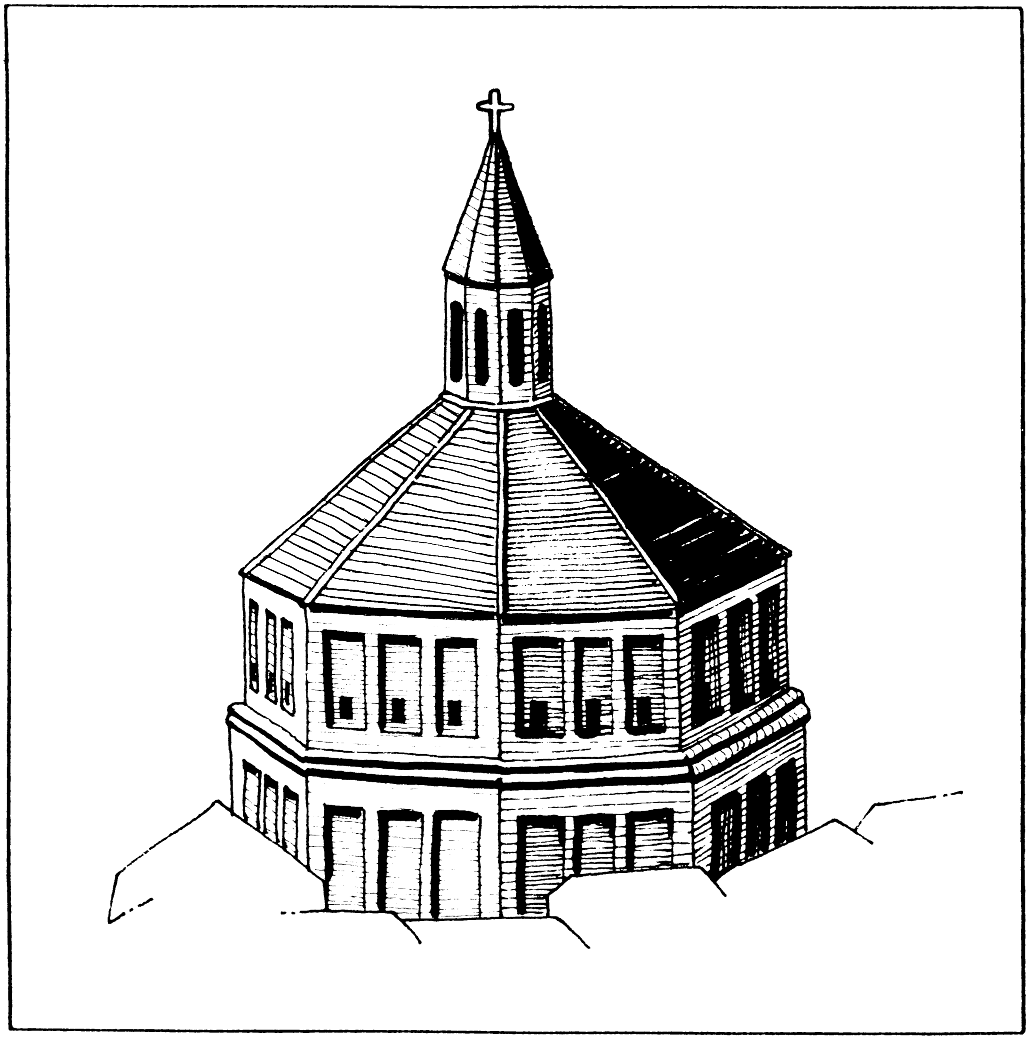
MATHEMATICS MAGAZINE



- Applied Mathematics as Social Contract
- Symmetry in Probability Distributions
- Some Famous Domes

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The aim of *Mathematics Magazine* is to provide lively and appealing mathematical exposition. This is not a research journal and, in general, the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for an article for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Articles on pedagogy alone, unaccompanied by interesting mathematics, are not suitable. Neither are articles consisting mainly of computer programs unless these are essential to the presentation of some good mathematics. Manuscripts on history are especially welcome, as are those showing relationships between various branches of mathematics and between mathematics and other disciplines.

The full statement of editorial policy appears in this *Magazine*, Vol. 54, pp. 44–45, and is available from the Editor. Manuscripts to be submitted should not be concurrently submitted to, accepted for publication by, nor published by another journal or publisher.

Send new manuscripts to: G. L. Alexanderson, Editor, Mathematics Magazine, Santa Clara University, Santa Clara, CA 95053. Manuscripts should be typewritten and double spaced and prepared in a style consistent with the format of *Mathematics Magazine*. Authors should submit the original and one copy and keep one copy. Illustrations should be carefully prepared on separate sheets in black ink, the original without lettering and two copies with lettering added.

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Two books, *The Mathematical Experience* and *Descartes' Dream*, written jointly with Reuben Hersh of the University of New Mexico, explore certain questions in the philosophy of mathematics, and have been translated into many languages. The former won the American Book Award for 1983.

Credit: Art for the note by LoBello provided by Dave Logothetti, Santa Clara University.

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ARTICLES

Applied Mathematics as Social Contract

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As compared to the medieval world or the world of antiquity, today's world is characterized as being scientific, technological, rational, and mathematized. By "rational" I mean that by an application of reason or of the formalized versions of reason found in mathematics, one attempts to understand the world and control the world. By "mathematized," I shall mean the employment of mathematical ideas or constructs, either in their theoretical form or in computer manifestations, to organize, to describe, to regulate and to foster our human activities. By adding the suffix "ized," I want to emphasize that it is humans who, consciously or unconsciously, are putting the mathematizations into place and who are affected by them. It is of vital importance to give some account of mathematics as a human institution, to arrive at an understanding of its operation and at a philosophy consonant with our experience with it, and on this basis to make recommendations for future mathematical education.

The pace of mathematization of the world has been accelerating. It makes an interesting exercise for young students to count how many numbers are found on the front page of the daily paper. The mere number of numbers is surprising, as well as the diversity and depth of the mathematics that underlies the numbers; and if one turns to the financial pages or the sports pages, one sees there natural language overwhelmed by digits and statistics. Computerization represents the effective means for the realization of current mathematizations as well as an independent driving force toward the installation of an increasing number of mathematizations.

Philosophies of Mathematics

Take any statement of mathematics such as 'two plus two equals four', or any more advanced statement. The common view is that such a statement is perfect in its precision and in its truth, is absolute in its objectivity, is universally interpretable, is eternally valid and expresses something that must be true in this world and in all possible worlds. What is mathematical is certain. This view, as it relates, for example, to the history of art and the utilization of mathematical perspective has been expressed by Sir Kenneth Clark ("Landscape into Art"): "The Florentines demanded more than an empirical or intuitive rendering of space. They demanded that art should be concerned with *certezza*, not with *opinioni*. Certezza can be established by mathematics."

The view that mathematics represents a *timeless* ideal of absolute truth and objectivity, and is even of nearly divine origin, is often called Platonist. It conflicts with the obvious fact that we humans have invented or discovered mathematics, that we have installed mathematics in a variety of places both in the arrangements of our daily lives and in our attempts to understand the physical world. In most cases, we can point to the individuals who did the inventing or made the discovery or the installation, citing names and dates. Platonism conflicts with the fact that mathematical applications are often *conventional* in the sense that mathematizations other than the ones installed are quite feasible (e.g., the decimal system). The applications are often *gratuitous*, in the sense that humans can and have lived out their lives without them (e.g., insurance or gambling schemes). They are *provisional* in the sense that alternative schemes are often installed which are claimed to do a better job (examples range all the way from tax legislation to Newtonian mechanics). Opposed to the Platonic view is the view that a mathematical experience combines the external world with our interpretation of it, via the particular structure of our brains and senses, and through our interaction with one another as communicating, reasoning beings organized into social groups.

The perception of mathematics as quasi-divine prevents us from seeing that we are surrounded by mathematics because we have extracted it out of unintellectualized space, quantity, pattern, arrangement, sequential order, change, and that as a consequence, mathematics has become a major modality by which we express our ideas about these matters. The conflicting views, as to whether mathematics exists independently of humans or whether it is a human phenomenon, and the emphasis that tradition has placed on the former view, leads us to shy away from studying the processes of mathematization, to shy away from asking embarrassing questions about this process: how do we install the mathematizations, why do we install them, what are they doing for us or to us, do we need them, do we want them, on what basis do we justify them? But the discussion of such questions is becoming increasingly important as the mathematical vision transforms our world, often in unforeseen ways, as it both sustains and binds us in its steady and unconscious operation. Mathematics creates a reality that characterizes our age.

The traditional philosophies of mathematics: platonism, logicism, formalism, intuitionism, in any of their varieties, assert that mathematics expresses precise, eternal relationships between atemporal mental objects. These philosophies are what Thomas Tymoczko has called "private" theories. In a private theory, there is one ideal mathematician at work, isolated from the rest of humanity and from the world, who creates or discovers mathematics by his own logico-intuitive processes. As Tymoczko points out, private theories of the philosophy of mathematics provide no account either for mathematical research as it is actually carried out, for the applications of mathematics as they actually come about, or for the teaching process as it actually unfolds. When teaching goes on under the banner of conventional philosophies of mathematics, it often becomes a formalist approach to mathematical education: "do this, do that, write this here and not there, punch this button, call in that program, apply this definition and that theorem." It stresses operations. It does not balance operations with an understanding of the nature or the consequences of the operations. It stresses syntactics at the expense of semantics, form at the expense of meaning. A fine place to read about this is in "L'age du Capitaine" by Stella Baruk, a mathematics supervisor in a French school system. Baruk writes

> From Pythagoras in antiquity to Bourbaki in our own day, there has been maintained a tradition of instruction-religion which sacrifices full understanding to the recitation of formal and ritual catechisms, which create docility and which simulate sense. All this has gone on while the High Priests of the subject laugh in their corners.

How many university lecturers, discoursing on numbers, say, allow themselves to discuss where they think numbers come from, what is one's intuition about them, how number concepts have changed, what applications they have elicited, what have been the pressures exerted by applications, how we are to interpret the consequences of these applications, what is the poetry of numbers or their drama or their mysticism, why there can be no complete or final understanding of them? How many lecturers would take time to discuss the question put by Bertrand Russell in a relaxed moment: "What is the Pythagorean power by which number holds sway above the flux?"

Opposed to "private" theories, there are "public" theories of the philosophy of mathematics in which the teaching process is of central importance. Several writers in the past half century have been constructing public theories, and I should like to add a few bricks to this growing edifice and to point out its relevance for the future of mathematical education.

Applied Mathematics as Social Contract

I shall emphasize the applications of mathematics to the social or humanistic areas though one can make a case for applications to scientific areas and indeed to pure mathematics itself. (See, e.g., [13].)

Today's world is full of mathematizations that were not here last year or ten years ago. There are other mathematizations which have been discarded (e.g., Ptolemaic astronomy, numerological interpretations of the cosmos, last years' tax laws). How do these mathematizations come about? How are they implemented, why are they accepted? Some are so new, for example, credit cards, that we can actually document their installation. Some are so ancient, e.g., numbers themselves, that the historical scenarios that have been written are largely speculative. Are mathematizations put in place by divine fiat or revelation? By a convention of Elders? By the insights of a gifted few? By an evolutionary process? By the forces of the market place or of biology? And once they are in place what keeps them there? Law? Compulsion? Inertia? Darwinian advantage? The development of a bureaucracy whose sole function it is to maintain the mathematization? The development of businesses whose function it is to create and sell the mathematization? Well, all of the above, at times, and more. But, for all the lavish attention that our historians of mathematics have paid to the evolution of ideas within mathematics itself, only token attention has been paid by scholars and teachers to the interrelationship between mathematics and society. A description of mathematics as a human institution would be complex indeed, and not be easily epitomized by a catch phrase or two.

The employment of mathematics in a social context is the imposition of a certain order, a certain type of organization. Government, as well, is a certain type of organization and order. Philosophers of the 17th and 18th century (Hobbes, Locke, Rousseau, Thomas Paine, etc.) put forward an idea, known as social contract, to explain the origin of government. Social contract is an act by which an agreed upon form of social organization is established. (Here I follow an article by Michael Levin [9].) Prior to the contract there was supposedly a "state of nature." This was far from ideal. The object of the contract, as Rousseau put it, was "to find a form of association which will defend the person and goods of each member with the collective force of all, and under which each individual, while counting himself with the others, obeys no one but himself, and remains as free as before." In this way, one may improve on a life which, as Hobbes put it in a famous sentence, was "solitary, poor, nasty, brutish, and short." The contract itself, whether oral or written, was almost thought of as having been entered into at a definite time and place. Old Testament history, with its covenants between God and Noah, Abraham, Moses, the Children of Israel, was clearly in the minds of contract theorists. In the United States, political thinking has often been in terms of contracts, as in the Mayflower Compact, the Constitutions of the United States and of the individual states, the establishment of the United Nations in San Francisco in 1945, and periodic proposals for constitutional amendments and reform.

It was generally assumed by the contract theorists that "Human society and government are the work of man constructed according to human will even if sometimes operating under divine guidance," that "man is a free agent, rather than a being totally determined by external forces," and that society and government are based on mutual agreement rather than on force. (See [**9**].)

The acceptability of social contract as a historical explanation hardly lasted till the 19th century, even if political contracts continued to be entered into as instances of democratic polity. It is an instructive exercise, I believe, in order to get a grasp on the relationship between society and mathematics, to take the outline of social contract just given and replace the words "government and society" by the word "mathematization." Though it is naive to think that most mathematizations came about by formal contracts, the "contract" metaphor is a useful phrase to designate the interplay between people and their mathematics and to make the point that mathematizations are the work of man, constructed according to human will, even if operating under a guidance which may be termed divine or logical or experimental according to one's philosophic predilection.

A number of authors, some writing about theology, and others about political or economic processes, have pointed out that contracts are continuously entered into, broken, and reestablished. I believe the same is the case for mathematizations. Consider, for example, insurance. This is one of the great mathematizations currently in place, and I personally, without adequate coverage, would consider myself naked to the world. Yet I am free to throw away my insurance policies. Consider the riders that insurance companies send me, unilaterally abridging their previous agreements. Consider also that in a litigious age, with a populace abetted by eager lawyers and unthinking juries, what appears as the 'natural' stability of the averages upon which possibility of insurance is based, emerges, on deeper analysis, to incorporate the willingness of the community to adjust its affairs in such a way that the averages are maintained. The possibilities of insurance can be destroyed by our own actions.

Another example that displays the relationship between mathematics, experience, and law is the highway speed limit in the United States. Before the gas shortage in 1974, the limit was 65 miles per hour. In 1974, the speed limit was reduced to 55 miles per hour in order to conserve gasoline. As a side effect, it was found that the number of highway accidents was reduced significantly. Now (1988) the gas shortage is over, and there is pressure to raise the legal speed limit. Society must decide what price it is willing to pay for what some see as the convenience or the thrill of higher speeds. Here is mathematical contract at work.

The process of contract maintenance, renewal or reaffirmation, in all its complexities, is open to study and description. This is a proper part of applied mathematics and I shall argue that it should be a proper part of mathematical education.

Where is Knowledge Lodged?

There is an epistemological approach to the interplay between mathematics and society, and that is to look at the way society answers the question that heads this section. According to how we answer this question, we will mathematize differently and we will teach differently.

Where, then, is knowledge lodged? (Here I follow an article by Kenneth A. Bruffee [3].) In the pre-Cartesian age, knowledge was often thought to be lodged in the mind of God. Those who imparted knowledge authoritatively derived their authority from their closeness to the mind of God, evidence of this closeness was often taken to be the personal godliness of the authority.

In the post-Cartesian age knowledge was thought to be lodged in some loci that are above and beyond ourselves, such as sound reasoning or creative genius or in the 'object objectively known.'

A more recent view, connected perhaps with the names of Kuhn and Lakatos, is that knowledge is socially justified belief. In this view, knowledge is not located in the written word or in symbols of whatever kind. It is located in the community of practitioners. We do not create this knowledge as individuals but we do it as part of a belief community. Ordinary individuals gain knowledge by making contact with the community of experts. The teacher is a representative of the belief community.

In my view, knowledge as socially justified belief provides a fair description of how mathematical knowledge is legitimized, but we must keep clearly in mind that perceptions of what 'is,' theory formation, validation, and utilization, are all part of a dynamic and iterative process. Knowledge once thought to be absolute, indubitable, is now seen as provisional or even probabilistic. Science is seen as a search for error as much as it is a search for truth. Eternally valid knowledge may remain an ideal which we hold in our minds as a spur to inquiry. This view fits with the idea of applied mathematics as social contract, with the contractual arrangements being concluded, broken, and renegotiated in endless succession.

Another view of the locus of knowledge, not yet elevated to a philosophy, is that knowledge is located in the computer. One speaks of such things as 'artificial intelligence,' 'expert systems,' and more than one theoretical physicist has opined that all the essentials are now known (despite the fact that the same was asserted 100 years ago and 200 years ago) and that the computer can fill in the details and derive the consequences for the future.

Advocates of this view have asserted that while education is now teacher oriented, in the full bloom of the computer age, education will be knowledge oriented. These two contemporary views are not necessarily antithetical, provided we accommodate the computer into the community of experts, clarify whether 'belief' can reside in a computer, and decide whether mankind exists for the sake of the computers or vice versa.

Mathematical Education at a Higher Metalevel

A mathematized and computerized world brings with it many benefits and many dangers. It opens many avenues and closes many others. (I do not want to elaborate this point as I and my coauthor Reuben Hersh have done so in our book *Descartes' Dream*, as have numerous other authors.)

The benefits and dangers both derive from the fact that the mathematical/computational way of thinking is different from other ways. Philosopher and historian Sir Isaiah Berlin called attention to this divergence when he wrote "A person who lacks common intelligence can be a physicist of genius, but not even a mediocre historian." For the mathematical way to gain ascendancy over other modes is to create an imbalance in human life.

The benefits and dangers derive also from the fact that mathematics is a kind of language, and this language creates a milieu for thought that is hard to escape. It both sustains us and confines us. As George Steiner has written of natural language [**16**]: "The oppressive birthright is the language, the conventions of identification and perception. It is the established but customarily subconscious unargued constraints of awareness that enslave." One can assert as much for mathematics as a language. The subconscious modalities of mathematics and of its applications must be made clear, must be taught, watched, argued. Since we are all consumers of mathematics, and since we are both beneficiaries as well as victims, all mathematizations ought to be opened up in the public forums where ideas are debated. These debates ought to begin in the secondary school.

Discussions of changes in mathematics curricula generally center around (a) the specific mathematical topics to be taught, e.g., whether to teach the square root algorithm, or continued fractions or projective geometry or Boolean algebra, and if so, in what grade, and (b) the instructional approaches to the specific topics, e.g., should they be taught with proofs or without; from the concrete to the abstract or vice versa, what emphasis should be placed on formal manipulations and what on intuitive understanding; with computers or without; with open ended problems or with "plug and chug" drilling.

Because of widespread, almost universal computerization, with hand-held computers that carry out formal manipulations and computations of lower and higher mathematics rapidly and routinely, because also of the growing number of mathematizations, I should like to argue that mathematics instruction should, over the next generation, be *radically* changed. It should be moved up from subject-oriented instruction to instruction in what the mathematical structures and processes mean in their own terms and what they mean when they form a basis on which civilization conducts its affairs. The emphasis in mathematics instruction ought to be moved from the syntactic-logico component to the semantic component. To use programming jargon, it ought to be "popped up" a metalevel. If, as some computer scientists believe, instruction is to move from being teacher oriented to knowledge oriented—and I believe this would be disastrous—the way in which the role of the teacher can be preserved is for the teacher to become an interpreter and a critic of the mathematical processes and of the way these processes interact with knowledge as a database. Instruction in mathematics must enter an altogether new and revolutionary phase.

The Interpretive Component of Mathematical Education

Let me begin by asking the question: to what end do we teach mathematics? Over the millenia, answers have been given and they have differed. Some of them have been: we teach it for its own sake, because it is beautiful; we teach it because it reveals the divine, because it helps us think logically, because it is the language of science and helps us to understand and reveal the world; we teach mathematics because it helps our students to get a job either directly, in those areas of social or physical sciences that require mathematics, or indirectly, insofar as mathematics, through testing, acts as a social filter, admitting to certain professional possibilities those who can master the material. We teach it also to reproduce ourselves by producing future research mathematicians and mathematics teachers.

Ask the inverse question: what is it that we want students to learn? We may answer this by citing specific course contents. For example, we may say that we now want to emphasize discrete mathematics as opposed to continuous mathematics. Or that we want to develop a course in nonstandard analysis on tape so that joggers may learn about hyperreal numbers even as they run. Or, we may decide for ourselves what the characteristic, constitutive ingredients of mathematical thought are: space, quantity, deductive structures, algorithms, abstraction, generalization, etc., and simply assure that the student is fed these basic ingredients, like vitamins. All of these questions and answers have some validity, and tradeoffs must occur in laying out a curriculum.

Within an overcrowded mathematical arena with many new ideas competing for inclusion in a curriculum, I am asking for a substantial elevation in the awareness of the applications of mathematics that affect society and of the consequences of these applications. If formal computations and manipulations can be learned rapidly and performed routinely by computer, what purpose would be served by tedious drilling either by hand or by computer? On-the-job training is certainly called for, whether at the supermarket checkout counter or on the blackboards of a hi-tech development company. If mathematics is a language, it is time to put an end to overconcentration on its grammar and to study the "literature" that mathematics has created and to interpret that literature. If mathematics is a logico-mechanism of a sort, then just as only a very few of us learn how to construct an automobile carburetor, but all of us take instruction in driving, so we must teach how to "drive" mathematically and *to interpret what it means when we have been driven mathematically in a certain manner.*

What does it mean when we are asked to create sex-free insurance pools? What are the consequences when people are admitted to or excluded from a program on a basis of numerical criteria? How does one assess a statement that procedure A is usually effective in dealing with medical condition B? What does it mean when a mathematical criterion is employed to judge the quality of prose or the comprehensibility of a poem or to create music in a programmatic way? What are the consequences of a computer program whose output is automatic military retaliation? The list of questions that need discussion is endless. Each mathematization-computerization requires explanation and interpretation and assessment. None of these things is now discussed in mathematics courses in the concrete form that confronts the public. If a teacher were inclined to do so, the reaction from his colleagues would probably be "Well, that is not mathematics. That is applied mathematics or that is psychology or economics or social-anthropology or law or whatever." My answer would be: I am trying, little by little, to bring in discussions of this sort into my teaching. It is difficult but important.

If the claim were made, with justice, that these matters cannot be discussed intelligently without deep knowledge both of mathematics and of the particular area of the real world, then I would agree, and point out that this claim forces into the open the conflict between democracy and "expertocracy." (See e.g., [**11**].) This conflict has received considerable attention in areas such as medicine, defense, and technological pollution, but has hardly been discussed at the level of an underlying mathematical language. The tension between the two claims, that of democracy and of expertocracy, could be made more socially productive by an education which enables a wide public to arrive at deeper assessments, moving from daily experience toward the details of the particular mathematizations. While we must keep in mind certain basic mathematical material, we must also learn to develop mathematical 'street smarts' which enable us to form judgements in the absence of technical expertise. (Cf. [**11**].)

A philosophy of mathematics which is "public" and not "private" lends support to introducing this kind of material into the curriculum. The discussion of such curriculum changes will be assisted by the perception of the mathematical enterprise as a human experience with contractual elements; and by the realization that every civilized person practices and utilizes mathematics at some level, and thereby enters a certain knowledge and belief community.

Again, following Kenneth Bruffees' article [3] (with additions and modifications), I would like to suggest several lines of inquiry.

(a) Identify and describe the mathematical beliefs, constructions, practices that are now in place. Where and how is mathematics employed in real life?

(b) Describe the mathematical beliefs, constructs, and practices that have been justified by the community. What are justifiable and unjustifiable? What are the modes of justification?

(c) Describe the social dimensions of mathematical practice. What constitutes a knowledge community? What does the community of mathematicians think are the best examples that the past has to offer?

As part of both (a) and (b) one should add: describe the nature of the various methods of *prediction* and the bases upon which prediction can become prescription (i.e., policy).

This type of inquiry is rarely carried out for mathematics. For example, the concrete question of where such and such a piece of university mathematics is used in practical life and how widespread is its use, is seldom answered. Many claims are made in textbooks, but show me the real bottom line. It is important to know. How, in fact, would you define the bottom line?

The technical term for inquiries such as the above is 'hermeneutics.' This word is well established in theology, and in the last generation has been commonly employed in literary criticism. It means the principles or the lines along which explanation and interpretation are carried out. It is time that this word is given a mathematical context. Instruction in mathematics must enter a hermeneutic phase. This is the price that must be paid for the sudden, massive and revolutionary intrusion of mathematizations-computerizations into our daily lives.

Conclusion

Mathematics is a social practice. This practice must be made the object of description and interpretation. It is ill-advised to allow the practice to proceed blindly by "mindless market forces" or as the result of the private decisions of a cadre of experts. Mathematical education must find a proper vocabulary of description and interpretation so that we are enabled to live in a mathematized world and to contribute to this world with intelligence.

Acknowledgement. I wish to thank Professor Reuben Hersh for numerous suggestions.

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Hooray for Calculus

Words: Bruce Reznick, Bob McEliece, Hal Fredricksen
Music: Johnny Mercer (Hooray for Hollywood)

Hooray for calculus,
Old Newton's rootin' tootin' calculus.
The class where letting delta x near zero
can make a hero
of students. Teachers will say,
Just take that limit,
Be bright not dim, it's
likely to be finite and you're on your way.

Hooray for calculus,
The single variable calculus.
It's mathematics where each lady and gent
can find a tangent,
the straight line part of a curve.
Approximation
and integration
are easy for the student with a little verve.

So here's to calculus.
Three cheers for ordinary calculus.
And while we know at first our students hate it,
We still A rate it,
At least it's worth a B + .
Though it's not child's play,
With work they come to say
Hooray for calculus.

The Ubiquitous π

Some well-known and little-known appearances of π in a wide variety of problems.

DARIO CASTELLANOS
Area de Estudios de Postgrado
Universidad de Carabobo
Valencia, Venezuela

Part II

Section 10. Eugene Salamin's Algorithm

In 1976, Eugene Salamin of Stanford, California, published in *Mathematics of Computation* an ingenious, quadratically converging algorithm for the calculation of π [78]. Quadratically converging means that the number of significant figures doubles after each step. The method is an adaptation of an algorithm discovered by Gauss for the evaluation of elliptic integrals.

The complete elliptic integrals of the first and second kinds are defined by

$$K(k)=\int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}dt, \tag{10-1}$$

and

$$E(k)=\int_0^{\pi/2}(1-k^2\sin^2 t)^{1/2}dt. \tag{10-2}$$

If $k^2+k'^2=1$, then $K(k')=K'(k)$, and $E(k')=E'(k)$ are also elliptic integrals, and they satisfy Legendre's relation:

$$K(k)E'(k)+K'(k)E(k)-K(k)K'(k)=\pi/2. \tag{10-3}$$

If $\{a_n\}$ is a convergent sequence with limit L, and if there exists a constant C, such that

$$|a_{n+1}-L|\leqslant C|a_n-L|^2 \tag{10-4}$$

for all n, then the sequence $\{a_n\}$ is said to converge quadratically to L.

Consider a triple of positive numbers (a_0, b_0, c_0) satisfying $a_0^2=b_0^2+c_0^2$. We proceed to determine number triples $(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots, (a_N, b_N, c_N)$ according to the following scheme of arithmetic-geometric means:

$$\begin{array}{ccc}
a_0 & b_0 & c_0 \\
a_1=\frac{1}{2}(a_0+b_0) & b_1=(a_0b_0)^{1/2} & c_1=\frac{1}{2}(a_0-b_0) \\
a_2=\frac{1}{2}(a_1+b_1) & b_2=(a_1b_1)^{1/2} & c_2=\frac{1}{2}(a_1-b_1) \\
\cdots & \cdots & \cdots \\
a_N=\frac{1}{2}(a_{N-1}+b_{N-1}) & b_N=(a_{N-1}b_{N-1})^{1/2} & c_N=\frac{1}{2}(a_{N-1}-b_{N-1}).
\end{array}$$

As a consequence of the arithmetic-geometric mean inequality one has $a_n \geqslant a_{n+1} \geqslant b_{n+1} \geqslant b_n$ for all n. It follows that $\{a_n\}$ and $\{b_n\}$ converge to a common limit, usually denoted by $agm(a_0, b_0)$.

Because of the definition of the c's above one easily obtains

$$c_{n+1} = \frac{1}{2}(a_n - b_n) = \frac{c_n^2}{4a_{n+1}} \leqslant \frac{c_n^2}{4agm(a_0, b_0)}, \tag{10-5}$$

and we see that the sequence $\{c_n\}$ converges quadratically to zero.

In calculations one stops at the Nth step, when $a_N = b_N$, i.e., when $c_N = 0$ to the desired degree of accuracy.

With $a_0 = 1$, $b_0 = \cos\alpha$, $c_0 = \sin\alpha$, it can be shown that, to the desired degree of accuracy,

$$K(\alpha) = \frac{\pi}{2a_N}, \tag{10-6}$$

$$\frac{K(\alpha) - E(\alpha)}{K(\alpha)} = \frac{1}{2}\left[c_0^2 + 2c_1^2 + 2^2c_2^2 + \cdots + 2^N c_N^2\right]. \tag{10-7}$$

Also, with $a_0' = 1$, $b_0' = \sin\alpha$, $c_0' = \cos\alpha$,

$$K'(\alpha) = \frac{\pi}{2a_N'}, \tag{10-8}$$

$$\frac{K'(\alpha) - E'(\alpha)}{K'(\alpha)} = \frac{1}{2}\left[c_0'^2 + 2c_1'^2 + 2^2c_2'^2 + \cdots + 2^N c_N'^2\right]. \tag{10-9}$$

Salamin's idea was to evaluate each of the elliptic integrals in Legendre's relation (10-3) by Gauss' arithmetic-geometric mean, equations (10-6) to (10-9), and then solve for π!

With k and k' as defined before, i.e., $k^2 + k'^2 = 1$, then

$$\pi = \frac{4agm(1,k)agm(1,k')}{1 - \sum_{j=1}^{\infty} 2^j\left(c_j^2 + c_j'^2\right)}. \tag{10-10}$$

With the symmetric choice $k = k' = 2^{-1/2}$, (10-10) becomes

$$\pi = \frac{4\left(agm(1, 2^{-1/2})\right)^2}{1 - \sum_{j=1}^{\infty} 2^{j+1}c_j^2}. \tag{10-11}$$

It was shown by L. V. King that Legendre's relation (10-3) is equivalent to the Gauss-Salamin formula (10-11) and that each may be derived from the other [**41**].

An improvement of this algorithm has been found quite recently by J. M. Borwein and P. B. Borwein [**13**]. To understand the nature of their result let us proceed to define exponential convergence.

A sequence $\{a_n\}$ is said to converge *exponentially* to L if there exists a constant $C > 1$, such that, for all n,

$$|a_n - L| \leqslant C^{-2^n}.$$

Exponential convergence and quadratic convergence are closely related: quadratic convergence implies exponential convergence and both types of convergence guarantee that a_n and L will agree through the first $O(2^n)$ digits, adopting the convention that 0.9999...9 and 1.0000...0 agree through the required number of digits.

Borwein and Borwein's result is:

Consider the three-term iteration with initial values

$$\alpha_0 = \sqrt{2}, \qquad \beta_0 = 0, \qquad \pi_0 = 2 + \sqrt{2} \tag{10-12}$$

given by,

(i)
$$\alpha_{n+1} = \frac{1}{2}\left(\alpha_n^{1/2} + \alpha_n^{-1/2}\right), \tag{10-13}$$

(ii)
$$\beta_{n+1} = \alpha_n^{1/2}\left(\frac{\beta_n + 1}{\beta_n + \alpha_n}\right), \tag{10-14}$$

(iii)
$$\pi_{n+1} = \pi_n \beta_{n+1}\left(\frac{1 + \alpha_{n+1}}{1 + \beta_{n+1}}\right). \tag{10-15}$$

Then π_n converges exponentially to π and

$$|\pi_n - \pi| \leqslant \frac{1}{10^{2^n}}. \tag{10-16}$$

The algorithm used by Shanks and Wrench would require about 1,000,000 operations to compute 1,000,000 digits of π. The above algorithm performs this feat in 200 operations!

In 1983 two Japanese, Yoshiaki Tamura of the International Latitude Observatory at Mizusawa and Yasuma Kanada of the University of Tokyo, calculated π successively to $2^{21}(2{,}097{,}152)$, to $2^{22}(4{,}194{,}304)$ and to $2^{23}(8{,}388{,}608)$ decimal places. They used Salamin's algorithm on a HITAC M-28OH. The calculation of 2^{23} digits took 6.8 hours of computer time. Subsequently they went on to calculate $2^{24}(16{,}777{,}216)$ places. This was surpassed in January, 1986, by D. H. Bailey of the NASA Ames Research Center who used a Cray-2 super computer to calculate π to 29,360,000 decimal places. The computation was based on the algorithm of Borwein and Borwein. In 1987 Yasuma Kanada went back to the calculation. Using this time a new Nippon Electric Corporation SX-2 supercomputer with extra memory banks he carried the tail of π to a whopping 134,217,728 (2^{27}) decimal places. Kanada now plans to double this figure.

Computing π to a few thousand decimal places is now a popular device for testing a new computer or training new programmers. Philip J. Davis, in his book *The Lore of Large Numbers*, writes, *The mysterious and wonderful pi is reduced to a gargle that helps computing machines clear their throats.*

William James in *The Meaning of Truth*, has the statement, *The thousandth decimal of pi sleeps there, though no one may ever try to compute it.*

No number has received more individual attention than π, nor has any other number been calculated to as many decimal places as π. There exists, for reasons unknown, a strange fascination with this number.

There also exists a quadratically converging algorithm for the calculation of roots [1]. The square root of 2 has, accordingly, been calculated to one million decimal places.

The cube root of sixteen is known to 1000 decimal places.

The real root of Wallis' equation, $x^3 - 2x - 5 = 0$, is known to 4000 digits. This equation was chosen by Wallis to illustrate Newton's method for the numerical solution of equations. It has since served as a test for other methods of approximation.

Harry L. Nelson of the Lawrence Livermore National Laboratory calculated one million factorial, a number having 5,565,709 digits.

The first ten thousand Fibonacci numbers are known. The 10,000th term has more than 2000 digits.

The base of the natural logarithms e, is known to 125,000 decimal places. This is 235 times smaller, in number of digits, than the current value of π mentioned above!

The largest known prime number (Mersenne prime) is $2^{132049} - 1$, a number having 39,751 digits.

The number π is also a favorite playground for mental calculators. We mentioned Johann Martin Zacharias Dase's feat in the last section. Steven B. Smith, *The Great Mental Calculators: The Psychology, Methods, and Lives of Calculating Prodigies, Past and Present*, Columbia University Press, 1984, reports on Hans Eberstark, linguist and mental calculator, who has memorized 11,944 digits of π! This, the book reports, is not the record. The number of digits of π committed to human memory staggers the imagination: 20,000, according to the latest *Guinness Book of World Records*.

Philip J. Davis, of Brown University, mentioned the following anecdote in private communication to the author: Davis asked Daniel Shanks of the University of Maryland, who, it will be remembered, together with John Wrench, did the first

PI = 3.+

1415926535	8979323846	2643383279	5028841971	6939937510	5820974944	5923078164	0628620899	8628034825	3421170679
8214808651	3282306647	0938446095	5058223172	5359408128	4811174502	8410270193	8521105559	6446229489	5493038196
4428810975	6659334461	2847564823	3786783165	2712019091	4564856692	3460348610	4543266482	1339360726	0249141273
7245870066	0631558817	4881520920	9628292540	9171536436	7892590360	0113305305	4882046652	1384146951	9415116094
3305727036	5759591953	0921861173	8193261179	3105118548	0744623799	6274956735	1885752724	8912279381	8301194912
9833673362	4406566430	8602139494	6395224737	1907021798	6094370277	0539217176	2931767523	8467481846	7669405132
0005681271	4526356082	7785771342	7577896091	7363717872	1468440901	2249534301	4654958537	1050792279	6892589235
4201995611	2129021960	8640344181	5981362977	4771309960	5187072113	4999999837	2978049951	0597317328	1609631859
5024459455	3469083026	4252230825	3344685035	2619311881	7101000313	7838752886	5875332083	8142061717	7669147303
5982534904	2875546873	1159562863	8823537875	9375195778	1857780532	1712268066	1300192787	6611195909	2164201989

3809525720	1065485863	2788659361	5338182796	8230301952	0353018529	6899577362	2599413891	2497217752	8347913151
5574857242	4541506959	5082953311	6861727855	8890750983	8175463746	4939319255	0604009277	0167113900	9848824012
8583616035	6370766010	4710181942	9555961989	4676783744	9448255379	7747268471	0404753464	6208046684	2590694912
9331367702	8989152104	7521620569	6602405803	8150193511	2533824300	3558764024	7496473263	9141992726	0426992279
6782354781	6360093417	2164121992	4586315030	2861829745	5570674983	8505494588	5869269956	9092721079	7509302955
3211653449	8720275596	0236480665	4991198818	3479775356	6369807426	5425278625	5181841757	4672890977	7727938000
8164706001	6145249192	1732172147	7235014144	1973568548	1613611573	5255213347	5741849468	4385233239	0739414333
4547762416	8625189835	6948556209	9219222184	2725502542	5688767179	0494601653	4668049886	2723279178	6085784383
8279679766	8145410095	3883786360	9506800642	2512520511	7392984896	0841284886	2694560424	1965285022	2106611863
0674427862	2039194945	0471237137	8696095636	4371917287	4677646575	7396241389	0865832645	9958133904	7802759009

9465764078	9512694683	9835259570	9825822620	5224894077	2671947826	8482601476	9909026401	3639443745	5305068203
4962524517	4939965143	1429809190	6592509372	2169646151	5709858387	4105978859	5977297549	8930161753	9284681382
6868386894	2774155991	8559252459	5395943104	9972524680	8459872736	4469584865	3836736222	6260991246	0805124388
4390451244	1365497627	8079771569	1435997700	1296160894	4169486855	5848406353	4220722258	2848864815	8456028506
0168427394	5226746767	8895252138	5225499546	6672782398	6456596116	3548862305	7745649803	5593634568	1743241125
1507606947	9451096596	0940252288	7971089314	5669136867	2287489405	6010150330	8617928680	9208747609	1782493858
9009714909	6759852613	6554978189	3129784821	6829989487	2265880485	7564014270	4775551323	7964145152	3746234364
5428584447	9526586782	1051141354	7357395231	1342716610	2135969536	2314429524	8493718711	0145765403	5902799344
0374200731	0578539062	1983874478	0847848968	3321445713	8687519435	0643021845	3191048481	0053706146	8067491927
8191197939	9520614196	6342875444	0643745123	7181921799	9839101591	9561814675	1426912397	4894090718	6494231961

5679452080	9514655022	5231603881	9301420937	6213785595	6638937787	0830390697	9207734672	2182562599	6615014215
0306803844	7734549202	6054146659	2520149744	2850732518	6660021324	3408819071	0486331734	6496514539	0579626856
1005508106	6587969981	6357473638	4052571459	1028970641	4011097120	6280439039	7595156771	5770042033	7869936007
2305587631	7635942187	3125147120	5329281918	2618612586	7321579198	4148488291	6447060957	5270695722	0917567116
7229109816	9091528017	3506712748	5832228718	3520935396	5725121083	5791513698	8209144421	0067510334	6711031412
6711136990	8658516398	3150197016	5151168517	1437657618	3515565088	4909989859	9823873455	2833163550	7647918535
8932261854	8963213293	3089857064	2046752590	7091548141	6549859461	6371802709	8199430992	4488957571	2828905923
2332609729	9712084433	5732654893	8239119325	9746366730	5836041428	1388303203	8249037589	8524374417	0291327656
1809377344	4030707469	2112019130	2033038019	7621101100	4492932151	6084244485	9637669838	9522868478	3123552658
2131449576	8572624334	4189303968	6426243410	7732269780	2807318915	4411010446	8232527162	0105265227	2111660396

6655730925	4711055785	3763466820	6531098965	2691862056	4769312570	5863566201	8558100729	3606598764	8611791045
3348850346	1136576867	5324944166	8039626579	7877185560	8455296541	2665408530	6143444318	5867697514	5661406800
7002378776	5913440171	2749470420	5622305389	9456131407	1127000407	8547332699	3908145466	4645880797	2708266830
6343285878	5698305235	8089330657	5740679545	7163775254	2021149557	6158140025	0126228594	1302164715	5097925923
0990796547	3761255176	5675135751	7829666454	7791745011	2996148903	0463994713	2962107340	4375189573	5961458901
9389713111	7904297828	5647503203	1986915140	2870808599	0480109412	1472213179	4764777262	2414254854	5403321571
8530614228	8137585043	0633217518	2979866223	7172159160	7716692547	4873898665	4949450114	6540628433	6639379003
9769265672	1463853067	3609657120	9180763832	7166416274	8888007869	2560290228	4721040317	2118608204	1900042296
6171196377	9213375751	1495950156	6049631862	9472654736	4252308177	0367515906	7350235072	8354056704	0386743513
6222247715	8915049530	9844489333	0963408780	7693259939	7805419341	4473774418	4263129860	8099888687	4132604721

Courtesy: Mathematics of Computation, Vol. XVI, No. 77, January 1962

π to 5,000 Decimals. From Daniel Shanks and John W. Wrench, Jr.

calculation of π with 100,000 places in 1961, the question of how many operations were needed to calculate π to n places? On the basis of his answer, Shanks predicted that mankind would never see one billion digits of π. One billion digits of π is slightly less than $2^{30} = 1{,}073{,}741{,}824$, and corresponds to 30 iterations of Salamin's formula. People have done the first 27 iterations. Will mankind never make the next three iterations, or will Shanks' prediction fall with the Chinese proverb to the effect that *it is silly to make predictions, especially with regard to the future*?

Section 11. Mnemonic Devices for π

Almost every language has some mnemonic device to recall the digits of π. The number of letters in each word represents the successive digits of π. Some of the better known ones are:

How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics!

This gives 3.14159265358979 [**15**].

The following poem was published in 1879 in *Nouvelle Correspondence Mathématique*, Brussels, v. V, p. 449. The author is unknown.

Que j'aime à faire apprendre un nombre utile aux sages!
Immortel Archimède, artiste ingénieur
Qui de ton jugement peut priser la valeur?
Pour moi ton problème eut de pareils avantages.

The Germans came, of course, with the following lines,

Dir, o Held, o alter Philosoph, du Riesen Genie!
Wie viele Tausende bewundern Geister,
Himmlisch wie du und göttlich!
Noch reiner in Aeonen
Wird das uns strahlen,
Wie im lichten Morgenrot!

The author of this poem is also, fortunately, unknown. The last two poems give 30 decimal places of π each.

The Colombian poet R. Nieto Paris, according to V. E. Caro, *Los Números*, Bogotá, 1937, p. 159, is the author of the following lines,

Soy π lema y razón ingeniosa
De hombre sabio que serie preciosa
Valorando enunció magistral
Con mi ley singular bien medido
El grande orbe por fin reducido
Fue al sistema ordinario cabal.

This poem gives 31 decimal places of π.

Inspired by F. De Lagny's calculation of π with 126 decimal places in 1717, P. Decerf wrote a poem where ten letter words replace zeros and which reproduces De Lagny's result. As we mentioned in Section 8, there was a mistake in the 113th place giving a 7 instead of an 8. The poem, accordingly, ended as follows,

Dédoublera chaque élément antérieur,
Toujours de l'orbe calculée approchera,
Laquelle limite donne l'arc, le secteur
De cet inquiétant cercle, ennemi trop rebelle,
Professeur, enseignez son problème avec zele...

When the error was detected, through Georg von Vega's calculation, the verses were modified, changing the word *secteur* (seven letters) by *quadrant* (eight letters), and in order not to break the rhyme, *antérieur* was duly changed by *précédent*, both words having nine letters.

These mnemonic devices are included here mostly for historical interest. It is, in fact, much simpler, to memorize π with 30 digits, than to memorize the poem and count the letters.

In the July 1985 issue of *Scientific American* A. K. Dewdney asked readers to find an easily remembered extension of the mnemonic: "How I wish I could enumerate pi easily." Peter M. Brigham of Brighton, Massachusetts, responded: "How I wish I could enumerate pi easily, since all these (censored) mnemonics prevent recalling any of pi's sequence more simply." This gives 3.14159265358979323846.

The same column that included the above, in the October 1985 issue, included an unusual mnemonic devised by Rodney A. Brooks of Bethesda, Maryland. It assigns each digit of pi to a note on the diatonic scale. In the key of C, for example, $C=1$, $D=2$, and so on. Beside each note digit a time value appears in parentheses. The value is a multiple of a 16th note's duration. Hence the notation is

$$3(1)1(1)4(1)1(1)5(5)9(1)2(1)6(1)5(6)3(1)5(1)8(5)9(1)$$

$$7(1)9(1)3(6)2(1)3(1)8(1)4(1)6(4)2(2)6(6)4(2)3(12).$$

The time signature is $4/4$ and the first bar line occurs between the fourth and fifth notes.

And finally, the Greek Linos J. Jacovides of Birmingham, Michigan, sent the following mnemonic:

ἀεὶ ο θεὸσ ὁ μέγασ γεωμετρεῖ τὸ κύκλου μῆκοσ ἶνα ὁρίοῃ διαμέτρῳ

("The almighty God plays with geometry in order to define the circumference of the circle in terms of its diameter.")

The author would very much enjoy hearing from readers who know mnemonic devices in languages other than the ones he has quoted.

Section 12. π and the Theory of Numbers

If $n \geqslant 1$ the Euler *totient function* $\phi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n, that is, it represents the number of numbers of the sequence $0, 1, \ldots, n-1$, which are relatively prime to n.

Examples are:

$$\phi(1)=1, \quad \phi(2)=1, \quad \phi(3)=2, \quad \phi(4)=2, \quad \phi(5)=4, \quad \phi(6)=2.$$

If we write n as a product of powers of primes,

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \tag{12-1}$$

then it can be shown that [4]

$$\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right), \tag{12-2}$$

or also

$$\phi(n)=p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)\cdots p_k^{\alpha_k-1}(p_k-1). \tag{12-3}$$

For instance,

$$\phi(120)=\phi(2^3\cdot 3\cdot 5)=2^2(2-1)3^0(3-1)5^0(5-1)=4\cdot 1\cdot 1\cdot 2\cdot 1\cdot 4=32.$$

Michael Keith noticed the following interesting fact [**40**]:

$$\phi(666)=216=6\cdot 6\cdot 6.$$

Many curious properties of the apocalyptic 666 have been discovered. For instance,

$$2^2+3^2+5^2+7^2+11^2+13^2+17^2=666,$$

found by Elvin J. Lee and which gives 666 as the sum of the squares of the first seven primes.

$$1^6-2^6+3^6=666.$$

$$6+6+6+6^3+6^3+6^3=666,$$

giving it as the sum of its digits plus the cubes of its digits.

$$(6+6+6)^2+(6+6+6)^2+6+6+6=666.$$

$$1^3+2^3+3^3+4^3+5^3+6^3+5^3+4^3+3^3+2^3+1^3=666.$$

$$e^{13/2}-\pi+4-(1/2)e^{-3\pi}\approx 666,$$

with an error of 4.101×10^{-8}.

The last three results are due to the author.

Monte Zerger noticed that the first nine digits of π arranged in triads

314 159 265

have the property that 159 and 265 form a Pythagorean triplet with 212 [**93**]:

$$159^2+212^2=265^2.$$

He goes on to point out that the triplet $(159, 212, 265)$ is a multiple of the triplet $(3,4,5)$. The other member of this triplet, 212, is the denominator in a good rational approximation to π:

$$\frac{666}{212}=3.1415094\ldots. \tag{12-4}$$

The triad 212 occurs very infrequently in the first 10,000 digits of π. It appears for the first time beginning at the 712th place, and then it does not appear again until the 6558th place!

Michael Keith noticed also that beginning at the 666th place after the decimal point of π there appears the triad $343=7\cdot 7\cdot 7$.

The function $\phi(n)$ is a multiplicative function, that is,

$$\phi(n_1 n_2)=\phi(n_1)\phi(n_2). \tag{12-5}$$

The function $\phi(n)$ has many curious properties. It is easy to prove, for instance, that if d designates the divisors of n, then,

$$\sum_{d|n} \phi(d) = n.$$

For example,

$$\phi(12) + \phi(6) + \phi(4) + \phi(3) + \phi(2) + \phi(1) = 12.$$

It can be shown [4] that, for $x > 1$

$$\sum_{n \leqslant x} \phi(n) = \frac{1}{2\zeta(2)} x^2 + O(x \log x) = \frac{3}{\pi^2} x^2 + O(x \log x). \tag{12-6}$$

Points in the plane whose coordinates are integers are called lattice points. Formula (12-6) has an interesting application concerning the distribution of lattice points in the plane which are visible from the origin.

Two lattice points P and Q are said to be mutually visible if the line segment which joins them contains no lattice points other than the endpoints P and Q.

We proceed to show [4] that *two lattice points* (a, b) *and* (m, n) *are mutually visible if, and only if,* $a - m$ *and* $b - n$ *are relatively prime.*

It is clear that (a, b) and (m, n) are mutually visible if and only if $(a - m, b - n)$ is visible from the origin. Hence, it suffices to establish the theorem when $(m, n) = (0,0)$.

Assume (a, b) is visible from the origin, and let d be the greatest common divisor of a and b. We will prove that $d = 1$. If $d > 1$ then $a = da'$, $b = db'$ and the lattice point (a', b') is on the line segment joining $(0,0)$ to (a, b). This contradiction proves that $d = 1$.

Conversely, suppose that a and b are relatively prime. If a lattice point (a', b') is on the line segment joining $(0,0)$ to (a, b), then

$$a' = ta, \qquad b' = tb, \quad \text{where} \quad 0 < t < 1.$$

Hence t is rational, so $t = r/s$ where r, s are positive integers which are relatively prime. Thus

$$sa' = ar \quad \text{and} \quad sb' = br,$$

so s is divisible by ar, and s is divisible by br. But since s and r are relatively prime then s is divisible by a and s is divisible by b. Hence $s = 1$ since a and b are relatively prime. This contradicts the inequality $0 < t < 1$. Therefore the lattice point (a, b) is visible from the origin.

There are infinitely many lattice points visible from the origin. We wish to find their distribution in the plane.

Consider a square region in the plane defined by the inequalities

$$|x| \leqslant r, \qquad |y| \leqslant r.$$

Let $N(r)$ denote the number of lattice points in this square, and let $N'(r)$ denote the number which are visible from the origin. The quotient $N'(r)/N(r)$ measures the fraction of those lattice points in the square which are visible from the origin. We wish to show that this fraction tends to a limit as $r \to \infty$.

The eight lattice points nearest the origin are all visible from the origin. By symmetry, one can see that $N'(r)$ is equal to 8, plus 8 times the number of visible points in the region

$$\{(x,y): 2 \leqslant x \leqslant r, \quad 1 \leqslant y \leqslant x\}.$$

This number is

$$N'(r) = 8 + 8 \sum_{2 \leqslant n \leqslant r} \sum_{\substack{l \leqslant m < n \\ (m,n)=1}} 1 = 8 \sum_{1 \leqslant n \leqslant r} \phi(n).$$

Because of (12-6) we have

$$N'(r) = \frac{24}{\pi^2} r^2 + O(r \log r).$$

But since the total number of lattice points in the square is

$$N(r) = (2[r]+1)^2 = (2r + O(1))^2 = 4r^2 + O(r),$$

we have,

$$\frac{N'(r)}{N(r)} = \frac{\frac{24}{\pi^2} r^2 + O(r \log r)}{4r^2 + O(r)} = \frac{\frac{6}{\pi^2} + O\left(\frac{\log r}{r}\right)}{1 + O\left(\frac{1}{r}\right)}.$$

Hence as $r \to \infty$ we find $N'(r)/N(r) \to 6/\pi^2$ [4].

The above result is sometimes described by saying that a lattice point chosen at random has probability $6/\pi^2$ of being visible from the origin. Or, if two integers a and b are chosen at random, the probability that they are relatively prime is $6/\pi^2$ [4]. In more mundane language, the probability that two one dollar bills chosen at random will have serial numbers which are relatively prime is $6/\pi^2$.

The Farey series F_n of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus p/q belongs to F_n if

$$0 \leqslant p \leqslant q \leqslant n, \qquad (p,q) = 1;$$

the numbers 0 and 1 are included in the forms $0/1$ and $1/1$.

The Farey series of order 7, F_7, for instance, is given by the nineteen fractions:

$$0/7,\ 1/7,\ 1/6,\ 1/5,\ 1/4,\ 2/7,\ 1/3,\ 2/5,\ 3/7,\ 1/2,$$
$$4/7,\ 3/5,\ 2/3,\ 5/7,\ 3/4,\ 4/5,\ 5/6,\ 6/7,\ 7/7.$$

The characteristic properties of Farey series are expressed by the theorems [31]: *If p/q and p'/q' are two successive terms of F_n, then*

$$qp' - pq' = 1; \tag{12-7}$$

and

If p/q, p''/q'', and p'/q' are three successive terms of F_n, then

$$\frac{p''}{q''} = \frac{p+p'}{q+q'}. \tag{12-8}$$

These two properties can be shown to be equivalent [31].

The number of terms in the Farey series of order n is obtained as follows: Since the fractions are all in lowest terms, it follows that for a given denominator q, the number of numerators is the number of integers less than q and prime to it, namely $\phi(q)$. This reasoning may be used for any integer from 1 to n; consequently, the number of fractions in the Farey series of order n is equal to $1+\phi(1)+\phi(2)+\cdots+\phi(n)$, if we take into account the zero fraction. Because of (12-6) we have the result that *the number of terms in the Farey series of order n is approximately* $1+3n^2/\pi^2$.

This result can be used to approximate the number of terms in the Farey series of order n, for large n. For $n=100$, for instance, $3\cdot 100^2/\pi^2+1$ equals $3040.6355\ldots$, whereas the correct value is 3045.

It can be shown that the *sum* of the fractions in the Farey series of order n is half the number of terms in the series [57].

A delightful account of Farey series and their many properties is found in G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* [31].

An allied problem is that of determining the probability that a number should be *quadratfrei*. A number is said to be quadratfrei if it is not divisible by the square of any prime.

The probability that a number should be quadratfrei is $6/\pi^2$: more precisely, the number $Q(x)$ of quadratfrei numbers not exceeding x is [31],

$$Q(x)=\frac{6x}{\pi^2}+O(\sqrt{x}).$$

John Farey observed the property given by (12-8) while perusing some tables of decimal quotients compiled by a certain Henry Goodwyn. Hardy and Wright, *The Theory of Numbers*, have the following comment on John Farey: "The history of 'Farey series' is very curious." Both (12-7) and (12-8) "seem to have been stated and proved by Haros in 1802; see Dickson, *History*, i. 156. Farey did not publish anything on the subject until 1816, when he stated" (12-8) "in a note in the *Philosophical Magazine*. He gave no proof, and it is unlikely that he had found one, since he seems to have been at the best an indifferent mathematician."

"Cauchy, however, saw Farey's statement, and supplied the proof (*Exercices de mathématiques*, i. 114–16). Mathematicians generally have followed Cauchy's example in attributing the results to Farey, and the series will no doubt continue to bear his name."

"Farey has a notice of twenty lines in the *Dictionary of National Biography*, where he is described as a geologist. As a geologist he is forgotten, and his biographer does not mention the one thing in his life which survives."

Section 13. π and the Laws of Chance

The most famous of all problems where π is related to probability arguments is the one posed, and solved, by George Louis Leclerc, Comte de Buffon (1707–1788). His problem, proposed in 1777, is the following: Let a needle of length L be thrown at random onto a horizontal plane ruled with parallel straight lines spaced from each other by a distance d greater than L. What is the probability that the needle will intersect one of these lines? It is simple to show [81] that the required probability is,

$$\int_0^{2\pi}\frac{L|\cos\theta|}{d}\frac{d\theta}{2\pi}=\frac{2L}{\pi d}. \tag{13-1}$$

R. Wolf in Zürich (1850) arrived at the original idea of calculating this probability experimentally by repeatedly dropping a needle on a ruled surface, counting the hits, and then solving (13-1) for π, to obtain its value statistically. See *Handbuch der Astronomie*, p. 127; *Mitteilungen der Berner Naturf. Gesellschaft*, 1850, pp. 85 and

209. With a needle 36 mm long, with a spacing between lines equal to 45 mm, and with 5000 throws, Wolf found $\pi = 3.1596$. The Englishmen A. Smith (1855) and Fox (1864) repeated the experiment and found with 3200 throws and 1100 throws, respectively, values of 3.1553 and 3.1419 for π.

It is not difficult to calculate the probability of obtaining k decimal places of π in N throws [9]. For example, the probability of obtaining π correct to five decimal places in 3400 throws of the needle is 0.015, which shows very poor convergence.

Edward Kasner and James R. Newman, *Mathematics and the Imagination*, New York, 1940, p. 247, report on the Italian mathematician Lazzerini dropping the needle 3408 times and finding $\pi = 3.1415929$, which shows that either Lazzerini picked a fluctuation around the mean, i.e., he did not previously specify the size of his sample, or the man was telling a story. I think the latter.

With digital computers it is not difficult randomly to generate π. Consider, for instance, a quadrant of a circle of unit radius inscribed in a unit square with center at one corner. If we randomly generate the abscissa x and the ordinate y, restricting their values to lie between 0 and 1, then the probability that the values $x^2 + y^2$ fall in the quadrant is proportional to the area of the quadrant: $\pi/4$. Thus π can be found as,

$$\pi = 4 \lim \frac{\{\text{number of } (x, y) | 0 \leqslant x^2 + y^2 \leqslant 1\}}{\{\text{number of } (x, y) | 0 \leqslant x^2 + y^2 \leqslant 2\}}, \tag{13-2}$$

where lim is meant to imply that the number of trials increases indefinitely.

The program, called PINT, was proposed by A. K. Dewdney in April 1985 Computer Recreations section of *Scientific American*; it brought a great number of replies from readers. Between 40,500 and 41,000 trials we found a minimum of 3.14156 and a maximum of 3.14338. This technique can, of course, be applied to any integral whose value is π. This procedure falls in the general category of Monte Carlo methods.

Section 14. Stories and Anecdotes Connected with π

United Press, Los Angeles, June 8, 1948, released the following story: *Carl Savit, Professor of Mathematics at the California Institute of Technology, appeared in court in demand of one thousand dollars plus legal expenses against the Mottant Company of Chicago. He claimed that this sum had been offered by this company in a published advertisement to the person who would solve the following problems:*

1. *Square the circle*
2. *Duplicate a cube*
3. *Trisect an angle with ruler and compass*

Savit claims to have solved all three problems and that he presented the solutions to the company but had been unable to collect the prize.

The company answered the demand by arguing that the advertisement did not constitute a contract and its only purpose was to catch the eye of the public.

Savit went on to explain how he had solved the problems: squaring the circle by means of the known construction given by Specht[†] *in 1828; the duplication of the*

[†]C. G. Specht, *Journal de Crelle*, vol. 3, 1836.

cube as done by Apollonius of Perga and the trisection of the angle by the method of Archimedes.

This story is partially true. Carl Savit was a graduate assistant at Caltech, erroneously described by a newspaper reporter as Professor of Mathematics. As it happened, though, the company had not specified Euclidean construction in their advertisement, i.e., compass and unmarked ruler and a finite number of operations, so that Mr. Savit was correct in his intent.

Specht's squaring of the circle is an approximate one, based on the formula $13\sqrt{146}/50 = 3.141591953\ldots$. It is possible, of course, to square the circle if one allows nonalgebraic curves such as Hippias' quadratrix. An angle can also be trisected with ruler and compass by using Archimedes' non-Euclidean construction, and a cube can be duplicated by using Diocles' cissoid. The three problems are, though, unsolvable in the realm of Euclidean geometry.

I do not know if Mr. Savit collected the prize.

Even after Lindemann's proof of the transcendence of π in 1882, which shows that it is impossible to make an Euclidean construction of a linear segment of length π, there are countless stories of cranks and circle-squarers who each year try to convince the universities of their wisdom and the correctness of their solutions.

In 1801, a pamphlet appeared in Cologne, Germany, with the following title: *Definitive Solution of the Diameter of a Circle to its Circumference, or the Discovery of the Squaring of the Circle by Chrétien Lowestein, Architect.* His method consisted of applying a metal band to a large wooden quadrant of a circle. By this method he found, for the relation of the circumference to the diameter, the value 3.1426.

In the year 1897 the Indiana House of Representatives considered, and unanimously approved, a bill that attempted to legislate a new value for π. The bill had originated with a physician named Edwin J. Goodman, M.D., of Solitude, Posey County, Indiana who claimed to have squared the circle and offered this contribution as a gift for the sole use of the State of Indiana. The bill was entitled "A bill introducing a New Mathematical Truth", and it became House Bill No. 246. The "new mathematical truth" gave for π the value $16/\sqrt{3} = 9.2376\ldots$. It is anybody's guess what would have happened had this bill followed its normal course until it became law. Fortunately for the State of Indiana, Professor C. A. Waldo of the Mathematics Department of Purdue University was visiting the State Capitol to make sure the Academy appropriations were cared for, and was greatly surprised, to say the least, to find the House in the midst of a debate on a piece of mathematical legislation. Professor Waldo quickly advised the senators, and on its second reading on February 12, 1897, the Senate voted to postpone the further consideration of this bill indefinitely, and it has not been on the agenda since [8].

The *Grand Dictionnaire Universel de Larousse*, v. XIII, 1875, page 483, says that the problem of the squaring of the circle was so important in the 16th century that Charles V of Spain offered one thousand escudos to anyone who solved it. This statement is completely false. It originates in an apocryphal letter of Charles V to Rabelais fabricated by the impostor Denis Vrain-Lucas and published in the *Annuaire de l'Observatoire de Bruxelles* in 1867. This letter reads:

Charles Quint à maistre François Rabelais, docteur en toutes sciences et bonnes lettres.

Maistre Rabelais,

Vous qu'avez l'esprit fin et subtil, me pourriez-vous satisfaire? J'ay promis 1000 *escus à celuy qui trouvera la quadrature du cercle et nul mathématicien n'a pu résoudre ce problesme. J'ay pansé que vous qui etes ingénieux en toutes chose me satisfairiez et, si le faicte, forte recompense en recevrez. Que Dieu vous vienne en ayde.*

Le X septembre 1542.

Charles.

This letter is one of 27,345 letters forged by Vrain-Lucas between 1861 and 1869. Vrain-Lucas sold these letters to the great scholar Michel Chasles for 140,000 francs. Regarding this swindle, see Pierre Bouchardon, *Les Procès Burlesques*, Paris, 1929, pp. 85–142; also Alphonse Daudet, *L'Immortel.*

Augustus de Morgan tells the following anecdote in *Assorted Paradoxes*:

"More than thirty years ago I had a friend, now long gone, who was a mathematician, but not of the higher branches: he was, *inter alia*, thoroughly up in all that relates to mortality, life assurance, &c. One day, explaining to him how it should be ascertained what the chance is of the survivors of a large number of persons now alive lying between given limits of number at the end of a certain time, I came, of course upon the introduction of π, which I could only describe as the ratio of the circumference of a circle to its diameter. 'Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?'—'I cannot demonstrate it to you; but it is demonstrated.'—'Oh! stuff! I think you can prove anything with your differential calculus: figment, depend upon it.'"

In 1750 Henri Sullamer made the announcement in England of having found the quadrature of the circle which, he claimed, depended upon the apocalyptical *666*. Each two or three years he published a new pamphlet in which he tried to show the exactness of his solution or where he presented new solutions [19].

Section 15. Epilogue

The reader has witnessed the never ending fascination that seems to exist with the number π. This is a field of endeavor that has attracted some of the greatest minds of mankind, and in spite of which appears never to be exhausted. Even the studious pursuer of the many curious and fascinating properties which surround this number will forever meet new results and new algorithms related to "the mysterious and wonderful pi." If I have succeeded in awakening an interest in the reader who is not, necessarily, a specialist in the field, then my aim will have been achieved.

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When I was at Princeton in 1937–38, I attended Hermann Weyl's current literature seminar. Weyl gave us a list of papers and asked people to sign up for one to report on. One was Ahlfors's famous paper on Nevanlinna theory; since I had been through it the preceding year with Ahlfors, it didn't seem interesting for me to select that one. Rufus Oldenburger, who was an algebraist, did think it would be interesting, but he got stuck in the middle: he couldn't see how a certain sentence followed from the one preceding it, and he asked me for help. I wrote out a detailed derivation for him, in about 6 pages. He looked at it for a while, and then said accusingly, "In algebra we *prove* the theorems."

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NOTES

The Volumes and Centroids of Some Famous Domes

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I. Introduction

The mathematical tourist observes, during his adventures, a great many different surfaces that architects have used over the centuries as models for the vaults they must provide to cover the open spaces of their buildings. If one takes a right circular cylinder of radius r and height h and cuts it into two equal pieces by a plane through the axis, each half is called a *barrel*, *tunnel*, or *wagon vault*, famous examples of which are to be found over the *Cloaca Maxima*, or great sewer of the ancient Roman Forum ($r = 10'$, $h = 800'$; see [**14**, p. 957]), and Carlo Maderno's nave for St. Peter's Basilica, built beginning in 1607 at the command of Pope Paul V ($r = 90'$, $h = 320'$; see [**1**, p. 364, and the plan between pages 362 and 363]). Among hemispherical domes, the geometer will most admire that of *Hagia Sophia* in Constantinople, built during the reign of the Emperor Justinian (527–565); its radius is 53′ (cf. [**3**, p. 548], and [**14**, p. 958b]). Now everyone knows the volumes of the half-cylinder and hemisphere, and any student of calculus learns how to calculate the centroids of those solids; in this essay, we look into the shapes of some other famous domes, and calculate their volumes and centroids.

II. The Roman Vault

The *Roman*, *intersecting barrel*, *cross*, or *groin vault* is the upper half of the solid produced by the perpendicular intersection along their axes of two equal right circular cylinders of radius r. Visitors to the Roman *Piscina Mirabilis* (the "wonderful reservoir") at Baiae may admire a great many such vaults with $r = 6\frac{1}{2}'$ [**14**, p. 959b] produced by the intersection of six aisles with barrel vaults with twelve other aisles with equal barrel vaults. The Roman vaults of the *tepidarium* (moderately heated room) of the Baths of Diocletian in Rome, constructed in 305 and transformed by Michelangelo into the nave (now the transept) of the Church of Santa Maria degli Angeli, have $r = 40'$ ([**14**], for an illustration, see [**12**, Fig. 1, Plate 142]); the resulting span is twice that of an English cathedral. The famous twin towers of the Benedictine Monastery at Einsiedeln, Switzerland, which were erected between 1704 and 1719, are crowned with cupolas inspired by the *Roman Vault*; these cupolas are not perfect Roman Vaults, though they appear to be so from below. They are built upon a square base about 15 meters by 15 meters. (See the plan in [**5**, p. 109].)

Let r be the common radius of the two cylinders whose intersection produces the vault. Suppose one of the cylinders has the x-axis for its axis, and the other the y-axis.

Einsiedeln Abbey

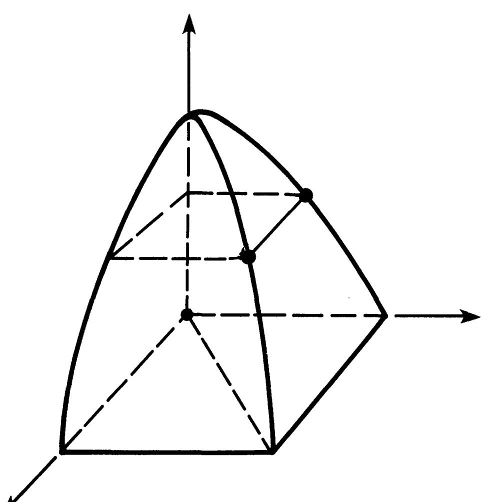
The vault then sits on the x-y plane. Consider the quarter of it that lies in the first octant. The plane sections are squares, and the side of the square at height z has length s, where $r^2 = z^2 + s^2$. The volume of the quarter-vault and its moment about the xy-plane are, therefore, given by

$$V = \int_0^r r^2 - z^2\,dz = \left(r^2 z - \frac{z^3}{3}\right)\Big|_0^r = 2r^3/3.$$

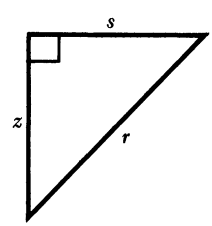
$$M_{xy} = \int_0^r z(r^2 - z^2)\,dz = \left(\frac{1}{2}r^2 z^2 - \frac{1}{4}z^4\right)\Big|_0^r = r^4/4.$$

Since $\bar{z}$ is the same for the quarter-vault and the whole vault, we have

$$\bar{z} = \frac{r^4/4}{2r^3/3} = \frac{3}{8}r.$$



(a)

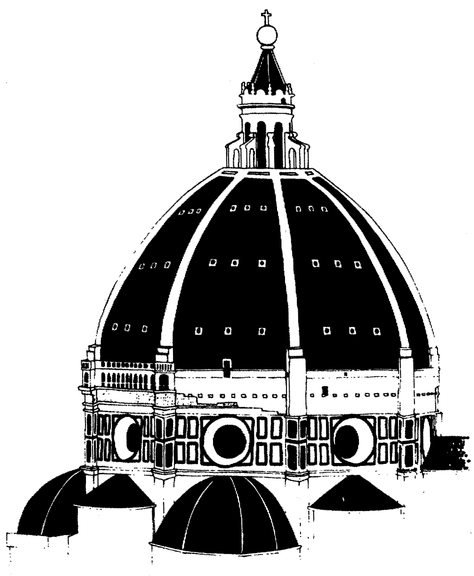


(b)

The volume of the vault is, of course, $8r^3/3$.

III. Intersections of Roman Vaults

Take a Roman vault $\mathscr{A}$ and rotate it 45° about its axis of symmetry to produce an equal Roman vault $\mathscr{A}'$. The solid of intersection $\mathscr{A} \cap \mathscr{A}'$ is also the upper half of the solid produced by intersecting four equal cylinders whose axes are the lines in the polar plane with equations $\theta = 0$, $\theta = \frac{1}{4}\pi$, $\theta = \frac{1}{2}\pi$, and $\theta = \frac{3}{4}\pi$. The plane sections are octagons and the ribs are elliptical arcs. Fine examples may be admired in the *baldacchini* (canopies) that cover the altar in the Chapel of St. Helen at the Church of the Aracoeli, Rome, and the high altar in the Basilica of Sant'Agnese fuori le Mura, also in Rome. Both are works of the seventeenth century, and the radius of the intersecting cylinders is in each case about 6′. (Cf. [**13**, p. 15], and [**10**, p. 146].) Perugino painted this kind of vault in the background of his earliest surviving work, "Christ Giving the Keys to St. Peter" in the Sistine Chapel [**7**, p. 69]. The most famous specimen, however, is the *inner* dome of Brunelleschi's cupola for the *Duomo* (cathedral) of Florence; the diameter of the circle in which the octagon at the base is inscribed is about 182 feet. (See [**6**, p. 142].) The inner dome and the outer one that covers it were constructed during 1420–1436; for an instructive picture (a vertical section) illustrating an inner and outer dome and the space between them, see [**9**, p. 22].



The Cathedral of Florence

The Roman Vault and Brunelleschi Inner Dome are special cases of cupolas produced by the intersection of equal cylinders. As the number of cylinders increases, the dome approaches a hemisphere. We now examine the general case. Let n be an even integer greater than 2. Take a cylinder of radius r and consider the solid of intersection of all the cylinders produced by rotating the given one about a fixed line perpendicular to its axis by $i(2\pi/n)$, $i = 1,2,3,\ldots,(1/2)n$. The solid is divided by the plane of the revolving axis into two equal pieces, each a dome whose plane sections parallel to the base are regular n-gons. If $n = 4$, we have a Roman Vault; if

$n = 8$, we have a Brunelleschi Inner Dome. In all cases the curves of intersection (the "ribs") are elliptical arcs. We proceed to calculate the volume of the dome and the height of its centroid above the base. The n-gonal slice at height z consists of n isosceles triangles; the altitude a of each triangle satisfies the equation

$$r^2 = z^2 + a^2,$$

so

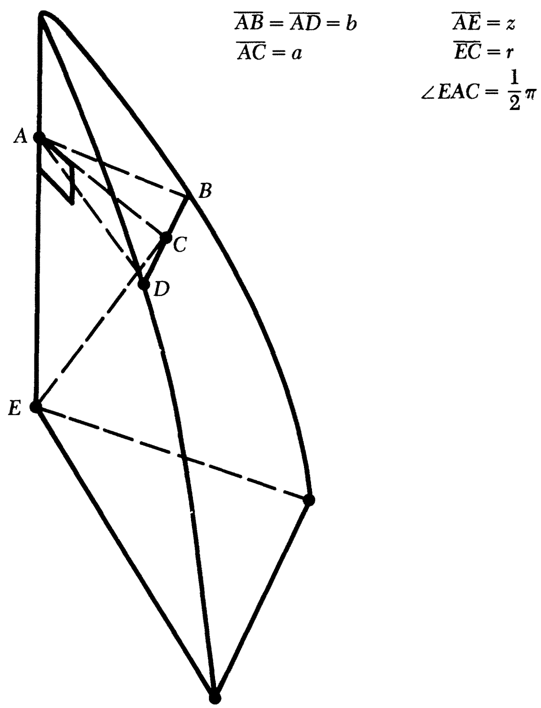
$$a = \sqrt{r^2 - z^2}.$$

If b is the length of the two sides that are equal, then, since

$$\frac{a}{b} = \cos\frac{\pi}{n},$$

we have

$$b = a\sec\frac{\pi}{n}.$$



(c)

Now it is easily proved that the area of an isosceles triangle whose equal sides are of length b and whose angle enclosed by these sides is ϕ is $\frac{1}{2}b^2\sin\phi$. The area of the triangle which we are considering is, therefore,

$$\frac{1}{2}a^2\sec^2\frac{\pi}{n}\sin\frac{2\pi}{n} = \frac{1}{2}(r^2 - z^2)\sec^2\frac{\pi}{n}\left(2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)$$
$$= (r^2 - z^2)\tan\frac{\pi}{n}.$$

The area of the n-gon at height z is, therefore, $(r^2 - z^2)n\tan(\pi/n)$, so the volume of the solid and its moment about its base are

$$V = \int_0^r \left(n\tan\frac{\pi}{n}\right)(r^2 - z^2)\,dz = \frac{2n}{3}\tan\frac{\pi}{n}r^3$$

$$M_{xy} = \int_0^r \left(n\tan\frac{\pi}{n}\right)z(r^2 - z^2)\,dz = \frac{1}{4}n\tan\frac{\pi}{n}r^4.$$

The height $\bar{z}$ of the centroid is, therefore,

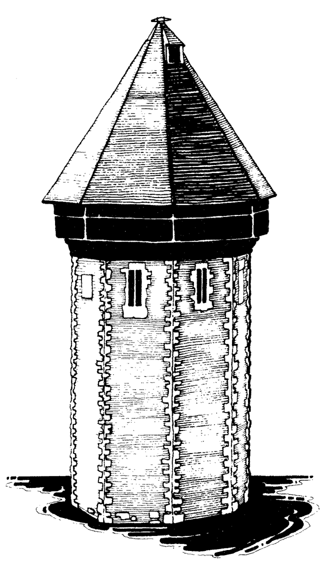
$$\frac{\frac{1}{4}n\tan\frac{\pi}{n}r^4}{\frac{2n}{3}\tan\frac{\pi}{n}r^3} = \frac{3}{8}r,$$

independent of n and the same as the value for a hemisphere of radius r.

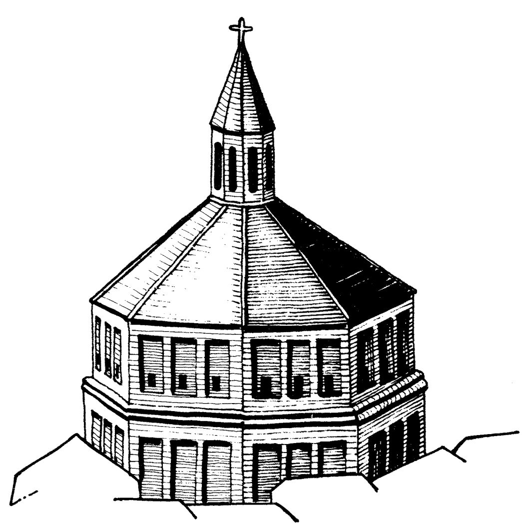
IV. The Baptistery, Florence

Consider an octagon inscribed in a circle of radius r, and pick a point P at height h above the center of the circle. Draw the lines from P to the points on the octagon. The solid thereby produced is the form of the dome of the *Baptistery of San Giovanni*, Florence; r is about 45′ ([**2**, p. 420]), h about 15′ ([**11**, p. 22]). It is also the shape of the water tower of Lucerne, Switzerland, the structure after which some incorrectly believe the city to be named (*lucerna* means lamp or lantern in Latin); the true etymology may be found in [**8**, p. 97]. The dimensions appear to be about 20 feet for r and 60 feet for h [**5**, p. 168].

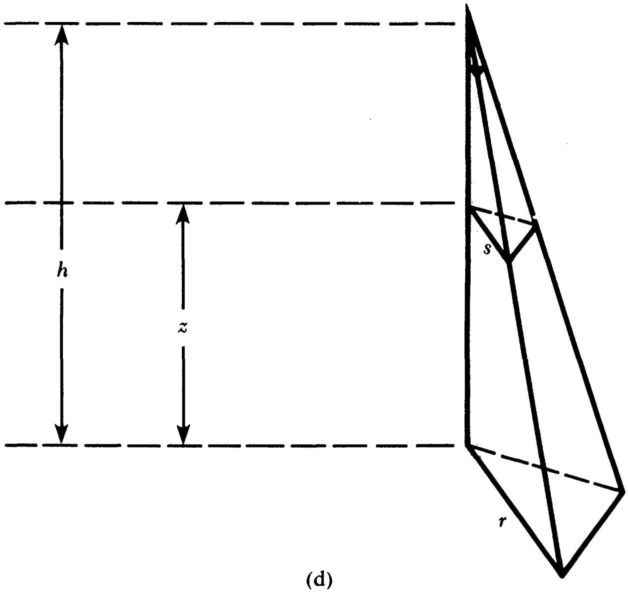
This dome is a special case of the following general solid. Consider a regular n-gon inscribed in a circle of radius r, and pick a point P at a height h above the center of the circle. Draw the lines from P to the points on the n-gon. The resulting solid is a dome whose plane sections are regular n-gons. As n approaches ∞, the dome



The Water Tower



San Giovanni



(d)

becomes more and more like a cone. We proceed to calculate its volume and the height of its centroid.

The slice at height z consists of n isosceles triangles; if each of the two equal sides is of length s, then we have

$$\frac{s}{r}=\frac{h-z}{h},$$

so

$$s=\frac{r(h-z)}{h}.$$

The area of the triangle is, therefore,

$$\frac{1}{2}\frac{r^2(h-z)^2}{h^2}\sin\frac{2\pi}{n}$$

and that of the whole slice

$$\frac{n}{2}\sin\frac{2\pi}{n}\frac{r^2(h-z)^2}{h^2}.$$

The volume of the dome and its moment about its base are, therefore, given by

$$V=\int_0^h\frac{nr^2\sin\frac{2\pi}{n}}{2h^2}(h^2-2hz+z^2)\,dz=\frac{nr^2\sin\frac{2\pi}{n}h}{6}.$$

$$\begin{aligned}M_{xy}&=\int_0^h\frac{nr^2\sin\frac{2\pi}{n}}{2h^2}z(h^2-2hz+z^2)\,dz\\&=\int_0^h\frac{nr^2\sin\frac{2\pi}{n}}{2h^2}(h^2z-2hz^2+z^3)\,dz=\frac{nr^2\sin\frac{2\pi}{n}h^2}{24}.\end{aligned}$$

The height $\bar{z}$ of the centroid above the base of the dome is, therefore,

$$\frac{\dfrac{nr^2\sin\dfrac{2\pi}{n}h^2}{24}}{\dfrac{nr^2\sin\dfrac{2\pi}{n}h}{6}}=\frac{1}{4}h,$$

independent of n and the same as that of the cone.

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On Some Determinant Inequalities and Cholesky Factorization

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An important, special version of Gauss elimination for symmetric matrices is the Cholesky algorithm [2]. For symmetric, positive definite matrices there is a very convenient factorization, which can be found without any need for pivoting or scaling. Systems of linear equations with positive definite, coefficient matrices are the most frequently occurring classes of structured linear systems, and they can be solved by first computing the Cholesky factorization.

Given a symmetric and positive definite matrix $A_{n\times n}$ (i.e., $A^T=A$ and $x^TAx>0$ for any nonzero $x\in R^n$), Cholesky's algorithm finds the following matrix factorization:

$A = LL^T$, where L is a lower triangular matrix whose elements are given recursively by the next relations:

(1) $l_{ij} = (a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk})/l_{jj}, \qquad j = 1, 2, \dots, i-1$

(2) $l_{ii} = (a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2)^{1/2}$.

Since A is positive definite, $l_{ii} > 0$, and from (2) we see that $l_{ii}^2 \leq a_{ii}$. The following well-known corollary can be easily derived.

COROLLARY. *If $A_{n \times n}$ is positive definite, then* $\det(A) \leq \prod_{i=1}^n a_{ii}$. *The equality occurs if and only if the matrix $A_{n \times n}$ is diagonal.*

Proof. Using Cholesky's factorization we have

$$\det(A) = \det(L)^2 = \prod_{i=1}^{n} l_{ii}^2 \leq \prod_{i=1}^{n} a_{ii}.$$

It is easy to see that equality is true if and only if A is diagonal.

The same idea can be extended to prove a more general inequality.

THEOREM. *If $M_{n \times n}$ is a matrix with column vectors $m_1, m_2, \dots m_n$, then*

$$|\det(M)| \leq \prod_{i=1}^{n} \|m_i\|,$$

where the norm used is the Euclidean norm.

Proof. If M is singular there is nothing to prove. When M is nonsingular the matrix $A = M^T M$ is symmetric, positive definite. Consider now the Cholesky factorization of A. Then $A = LL^T$, and using (2) we obtain

$$a_{ii} = \sum_{k=1}^{i} l_{ik}^2 = \|m_i\|^2.$$

Therefore, we have

$$(\det(M))^2 = \det(A) = (\det(L))^2 = \prod_{i=1}^{n} l_{ii}^2 \leq \prod_{i=1}^{n} a_{ii} = \prod_{i=1}^{n} \left(\sum_{k=1}^{i} l_{ik}^2 \right) = \prod_{i=1}^{n} \|m_i\|^2.$$

Equality holds if and only if $a_{ii} = l_{ii}$ or $l_{ij} = 0$, $i \neq j$. This means that the matrix $A = LL^T$ is diagonal or, equivalently, the columns of the original matrix M are orthogonal.

This is Hadamard's classical inequality. In geometric language it says that the volume of a parallelopiped, with edges the vectors $m_1, \dots, m_n$ in R^n, is less than or equal to the product of the lengths of the edges. Although standard proofs by induction or determinant techniques exist [1], [3], this proof has a particular simplicity.

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Brocard Points, Circulant Matrices, and Descartes' Folium

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In the flourishing days of triangle geometry, many special points were discovered and investigated. Apart from well-known points like the centroid (or median point), the orthocenter, and the circumcenter, 'new' triangle points were studied, points called the symmedian point (or point of Lemoine) and the points of Gergonne, Nagel, Torelli, and Brocard, to name but a few. There is little doubt in my mind that the Brocard points rank amongst the most interesting of these special points associated with the triangle. Although general interest has long since waned and results once regarded as important have sunk into oblivion, it might still be worth our while to revive some of the gems of triangle geometry. In the brilliant light of modern knowledge we might even discover new and interesting insights.

In the literature on Euclidean geometry some books can be singled out that deal exclusively with the geometry of the triangle and the circle. An excellent monograph is [**4**], and for those with a smattering of German [**3**] gives much information, too; [**5**] is of a more general nature, but this work also contains many pages devoted to the triangle and its associated points. Finally, the Brocard configuration is the singular topic of Emmerich's treatise [**2**], recommendable for its proverbial 'Gründlichkeit.'

In the rich field of Brocardian geometry, our attention shall be focused on the set of triangles equibrocardal to a given triangle (T). In order to explain the terminology, our first concern should be with the reader who wishes to be introduced to the Brocard points and the Brocard angle of a plane triangle.

Well then, given a triangle (T) with vertices A_1, A_2, and A_3, notation: $(T) = A_1A_2A_3$, the first (or positive) Brocard point of (T) is the unique point Ω such that the angles $\angle\Omega A_1A_2$, $\angle\Omega A_2A_3$, and $\angle\Omega A_3A_1$ are equal. The second (or negative)

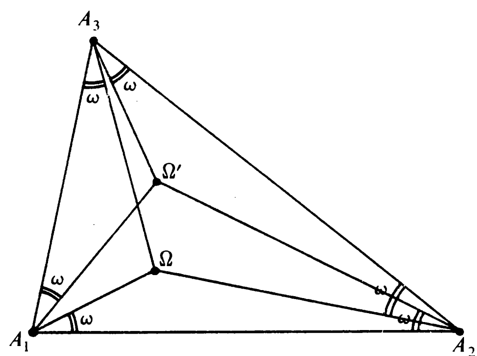

FIGURE 1.
The Brocard points Ω and Ω' of triangle $(T) = A_1A_2A_3$.

Brocard point Ω' of $(T) = A_1A_2A_3$ is the first Brocard point of triangle $(T)' = A_1A_3A_2$, which is obtained from (T) by changing its orientation (see FIGURE 1). As it happens, the angles $\omega := \angle\Omega A_1A_2 = \angle\Omega A_2A_3 = \angle\Omega A_3A_1$ and $\omega' := \angle\Omega' A_1A_3 = \angle\Omega' A_3A_2 = \angle\Omega' A_2A_1$ coincide; the common value is known as the Brocard angle of (T). These and other useful facts shall be discussed in the next section.

The main questions we shall be concerned with in this note are:

(i) How can one describe in a systematic way all plane triangles with Brocard angle equal to that of a given triangle (T)? Such triangles are known as equibrocardal.

(ii) If one restricts the positions of equibrocardal triangles in some natural way so as to avoid duplications by translation, rotation, similarity, etc., how can one describe the locus of the Brocard points of these triangles?

Answers to these questions shall be given in subsequent sections, but first we have to introduce some simple facts about the Brocard configuration.

Some Basic Facts

First suppose that the interior of triangle $(T) = A_1A_2A_3$ contains a point Ω such that $\angle\Omega A_1A_2 = \angle\Omega A_2A_3 = \angle\Omega A_3A_1$. Then the line joining A_2 and A_3 is tangent to the circle (c_2) through A_1, A_2, and Ω. This can be seen by observing that $\angle\Omega A_1A_2$ inscribed in the arc $A_2\Omega$ of (c_2) equals $\angle\Omega A_2A_3$ (see FIGURE 2). This means that Ω is a point common to three circles, each tangent to one side of (T) at different vertices and passing through a second vertex of (T). Conversely, it is not really difficult to see that the three circles (c_1), (c_2), and (c_3), where (c_i) is tangent to A_iA_{i+1} at A_i and passing through A_{i+2}, are concurrent. Here the indices i of A_i are taken modulo 3, which means that A_i and A_j are identical whenever $i \equiv j \pmod 3$. The point of intersection Ω of these circles is necessarily interior to (T). Clearly, Ω is completely determined by this construction.

Having established the existence and uniqueness of the Brocard points Ω and Ω', we turn our attention to an interesting analytical identity, which may serve as a defining expression for the Brocard angle ω. In order to convince ourselves of its validity, we need a little trigonometry.

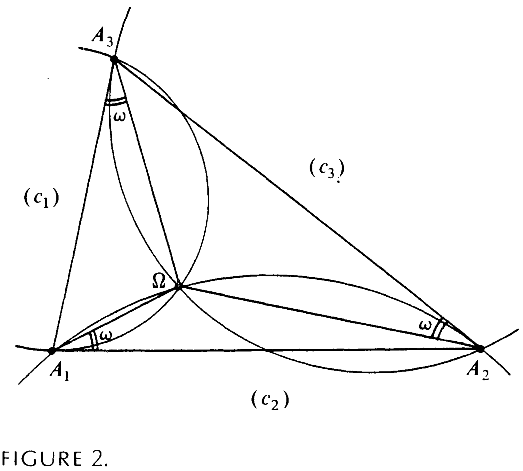

FIGURE 2.
Concurrent circles $(c_1), (c_2)$, and (c_3) meeting at Ω.

Let α_1, α_2, and α_3 denote the angles of (T) at the vertices A_1, A_2, and A_3 with opposite sides a_1, a_2, and a_3, respectively. Applying the rule of sines successively in the triangles $A_1A_3\Omega$, $A_2A_3\Omega$ and $A_1A_2A_3$ (see FIGURE 2) yields

$$A_3\Omega/\sin(\alpha_1-\omega)=a_2/\sin\alpha_1,$$
$$A_3\Omega/\sin\omega=a_1/\sin\alpha_3,$$

and

$$a_1/\sin\alpha_1=a_2/\sin\alpha_2,$$

respectively. Eliminating a_1, a_2 and $A_3\Omega$ from these expressions gives

$$\sin(\alpha_1-\omega)\sin\alpha_2\sin\alpha_3=\sin^2\alpha_1\sin\omega,$$

and dividing through by $\sin\alpha_1\sin\omega$, using $\alpha_1+\alpha_2+\alpha_3=\pi$, results in

$$(\cot\omega-\cot\alpha_1)\sin\alpha_2\sin\alpha_3=\sin(\alpha_2+\alpha_3).$$

The reader is invited to deduce the elegant identity

$$\cot\omega=\cot\alpha_1+\cot\alpha_2+\cot\alpha_3. \tag{1}$$

On squaring (1) and observing that, because $\alpha_1+\alpha_2+\alpha_3=\pi$,

$$\cot\alpha_1\cot\alpha_2+\cot\alpha_2\cot\alpha_3+\cot\alpha_3\cot\alpha_1=1,$$

we obtain the equivalent relation

$$\cot^2\omega=\cot^2\alpha_1+\cot^2\alpha_2+\cot^2\alpha_3+1,$$

which is easily rewritten as

$$1/\sin^2\omega=1/\sin^2\alpha_1+1/\sin^2\alpha_2+1/\sin^2\alpha_3. \tag{2}$$

From the obvious inequalities

$$0<\omega<\min(\alpha_1,\alpha_2,\alpha_3)<\pi/2,$$

it follows immediately that (1)—and hence also (2) by equivalence—uniquely determine ω. Also, by symmetry, we find that $\cot\omega=\cot\omega'$ which implies $\omega=\omega'$, an assertion made before but unproven so far.

Many pleasing relations between the Brocard angle ω and other triangle quantities can be established in a similar way. We refer to [2] and [6] for details. Particularly useful is the following relation between ω, the sides a_i, and the area Δ of (T):

$$4\Delta\cot\omega=a_1^2+a_2^2+a_3^2. \tag{3}$$

To prove this, we recall that

$$2\Delta=a_2a_3\sin\alpha_1.$$

By the rule of cosines in triangle $A_1A_2A_3$ we also have

$$a_1^2=a_2^2+a_3^2-2a_2a_3\cos\alpha_1.$$

Combining these expressions yields

$$4\Delta\cot\alpha_1=-a_1^2+a_2^2+a_3^2.$$

Because of symmetry, similar formulae exist for $\cot\alpha_2$ and $\cot\alpha_3$. Substitution into (1) of the expressions for $\cot\alpha_i$ thus obtained immediately gives (3).

In a later section we shall construct an analytic formula for the exact position of Ω in relation to the positions of the vertices A_1, A_2, and A_3 of the given triangle (T). The formulae (1), (2), and (3) play a significant part in that construction.

An obvious inequality for the Brocard angle ω—we mentioned it before—is

$$\omega < \min(\alpha_1, \alpha_2, \alpha_3).$$

One may ask instead for an absolute upper bound for ω, i.e., an upper bound independent of (T), and preferably one that is least, so that for each value below this bound a triangle exists with Brocard angle agreeing with that value. Again we shall use some trigonometry. From (1) we deduce

$$\begin{aligned}\sin(\alpha_1+\omega)/\sin\omega &= \sin\alpha_1\cot\omega+\cos\alpha_1\\ &= \sin\alpha_1(\cot\alpha_1+\cot\alpha_2+\cot\alpha_3)+\cos\alpha_1\\ &= \sin(\alpha_1+\alpha_2)/\sin\alpha_2+\sin(\alpha_1+\alpha_3)/\sin\alpha_3\\ &= \sin\alpha_3/\sin\alpha_2+\sin\alpha_2/\sin\alpha_3,\end{aligned}$$

so that, by the rule of sines in (T),

$$\sin(\alpha_1+\omega)/\sin\omega = a_3/a_2+a_2/a_3.$$

This shows that

$$\sin(\alpha_1+\omega)/\sin\omega \geqslant 2,$$

with equality if and only if $a_2=a_3$. Consequently

$$2\sin\omega \leqslant \sin(\alpha_1+\omega) \leqslant 1,$$

and hence

$$0<\omega\leqslant\pi/6, \tag{4}$$

with equality if and only if triangle (T) is equilateral.

When asking for a triangle (T), by construction or otherwise, with a prescribed Brocard angle $\omega\leqslant\pi/6$, one is actually asking for the possible values of the angles α_1, α_2 and α_3 of (T). In other words, one wants to find $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_i>0$, $\alpha_1+\alpha_2+\alpha_3=\pi$ and such that $\cot\alpha_1+\cot\alpha_2+\cot\alpha_3$ has a prescribed value $\geqslant\sqrt{3}=\cot(\pi/6)$.

Picking up our interrupted argument again, define the function

$$h(\delta)=\sin(\delta+\omega)-2\sin\omega \tag{5}$$

in the variable δ, where ω has a fixed value in the range $0<\omega\leqslant\pi/6$. Restricting δ to values between 0 and π, it is obvious that the function $h(\delta)$ has precisely two zeros δ_1 and δ_2. Clearly, $\delta_1+\delta_2=\pi-2\omega$. Moreover, observing that

$$\cot\delta+\cot\omega=\sin(\delta+\omega)/(\sin\delta\sin\omega)=2\sin\omega/(\sin\delta\sin\omega)=2/\sin\delta,$$

we see that the equation $h(\delta)=0$ may be rewritten as

$$\cot^2(\delta/2)-2\cot\omega\cot(\delta/2)+3=0.$$

Hence

$$\cot(\delta_1/2) = \cot\omega + \sqrt{\cot^2\omega - 3}$$

and (6)

$$\cot(\delta_2/2) = \cot\omega - \sqrt{\cot^2\omega - 3}\,.$$

Note that the condition (4) is necessary and sufficient for the existence of δ_1 and δ_2. Since

$$h(\alpha) = \sin(\alpha + \omega) - 2\sin\omega \geqslant 0$$

for any one of the angles α of a triangle (T) with Brocard angle ω, it follows immediately that

$$\delta_1 \leqslant \alpha \leqslant \delta_2. \tag{7}$$

Apparently, δ_1 and δ_2 give the minimal and maximal values respectively that any angle of a triangle with prescribed Brocard angle ω can possibly attain. Conversely, given $\omega \in (0, \pi/6]$, choose $\alpha_1 = \alpha$ satisfying (7). Then the expression

$$\sin(\alpha + \omega)/(2\sin\omega)$$

uniquely determines the ratio of the sides a_2 and a_3, provided we prescribe the sign of $a_2 - a_3$. The resulting triangles are all similar and are equibrocardal with Brocard angle ω.

FIGURE 3 below gives some values of ω and corresponding values of δ_1 and δ_2.

ω	5.00	10.00	15.00	20.00	25.00	27.50	29.00	29.50	29.75	30.00
δ_1	5.04	10.32	16.17	23.16	32.70	39.94	46.84	50.51	53.20	60.00
δ_2	164.96	149.68	133.83	116.84	97.30	85.06	75.16	70.49	67.30	60.00

FIGURE 3.
Minimal value δ_1 and maximal value δ_2 for the angles of triangles with prescribed Brocard angle ω measured in degrees.

The Neuberg Circles

Having set the scene, and preparations being complete, we can now embark on the investigation of the set of triangles equibrocardal to a given triangle (T).

To begin with, let us agree to the following restrictions, which can be made without losing generality. Usually, we shall only consider triangles with the same orientation as the given triangle $(T) = A_1A_2A_3$, i.e., the vertices are numbered counterclockwise. Further, it is clearly sufficient to choose only one representative from each class of directly similar triangles. Two triangles are called directly similar if the one is homothetic to the image of the other after a suitable translation and/or rotation. So, directly similar means similar, but orientation preserving.

The restrictions imposed so far are clarified by the correspondence between triangles and points $(\alpha_1, \alpha_2, \alpha_3)$ in 3-space, situated in the plane $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. In other words, each triangle, up to similarity, is given by an ordered 3-tuple of angles. In this way a restricted set of equibrocardal triangles with prescribed Brocard angle may be visualized as a closed curve in the plane $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ in 3-space. However the points $(\alpha_1, \alpha_2, \alpha_3)$, $(\alpha_3, \alpha_1, \alpha_2)$, and $(\alpha_2, \alpha_3, \alpha_1)$ correspond to directly similar triangles. So only one third of the ω-curve, namely the part contained in the shaded region (see FIGURE 4), corresponds to all triangles with the same Brocard angle as (T), but not directly similar to (T). In FIGURE 4 the position of triangle (T) of FIGURE 1 is shown as a point on the ω-curve. We also should make a positional choice, that is to say, to some extent we are free to prescribe the positions of the triangles in the Euclidean plane. This may be done in various ways. Two natural possibilities present themselves, namely, the position of one side could be fixed for all triangles, or they could be required to have a common point, like the centroid. Recall that the centroid of a triangle is the point at which the medians meet. From the last lines of the previous section it is clear that for any given $\omega \in (0, \pi/6]$, each triple (α_1, A_1, A_2), where α_1 is chosen in size between δ_1 and δ_2, and $A_1 \neq A_2$, uniquely determines the third vertex A_3 of triangle $(T) = A_1A_2A_3$ with Brocard angle ω and prescribed orientation. Thus, if we restrict the positions of the equibrocardal triangles by fixing their base A_1A_2—or any other side—it should be possible to describe the locus of the third vertex A_3. In FIGURE 5 triangle (T) has Brocard angle ω, and N lies on the perpendicular bisector of A_1A_2 at a distance $l = \frac{1}{2}a_3\cot\omega$ from the middle M of the base A_1A_2, where a_3 is the length of A_1A_2. Further, we denote by m the length of the median from A_3 and by ϕ the angle $\angle A_3MN$. Finally, r designates the length of the third side of triangle A_3MN. We intend to prove that r only depends on ω and a_3, to be precise

$$2r = a_3\sqrt{\cot^2\omega - 3}\,. \tag{8}$$

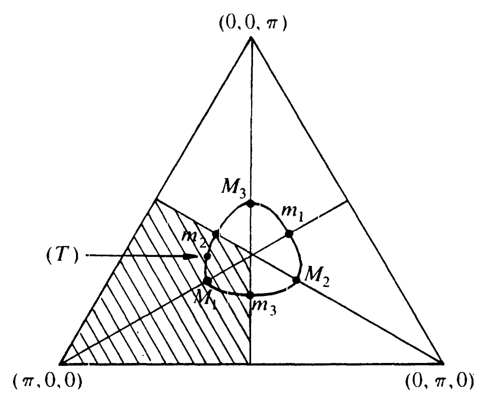

FIGURE 4.
ω-curve ($\omega \approx 26.9$ deg).

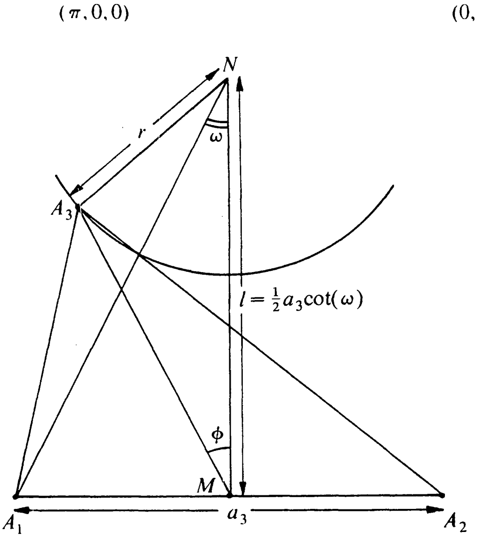

FIGURE 5.
Locus of A_3 for fixed ω and base A_1A_2 of length a_3.

We leave it to the reader to verify that

$$4m^2 = 2a_1^2 + 2a_2^2 - a_3^2 \quad \text{and} \quad 2\Delta = ma_3\cos\phi.$$

Here, as usual, Δ denotes the area of triangle $A_1A_2A_3$. By the law of cosines in triangle A_3MN we obtain

$$\begin{aligned} r^2 &= l^2 + m^2 - 2lm\cos\phi = l^2 + m^2 - 2\Delta\cot\omega \\ &= \frac{1}{4}a_3^2\cot^2\omega + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2 - \frac{1}{4}a_3^2 - \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2\right), \end{aligned}$$

according to (1). Consequently, $4r^2 = a_3^2(\cot^2\omega - 3)$ as required.

Since N is fixed—indeed, l depends on a_3 and ω only—the geometric interpretation of this result is that A_3 describes a circle with centre N and radius r, given by (8). This circle and its two associates, each obtained by fixing one side of the given triangle, are called the Neuberg circles, after their discoverer.

All triangles having their base of length b, their Brocard angle ω as well as their orientation in common with a given triangle (T), have their third vertex on a so-called Neuberg circle of diameter $2r = b\sqrt{\cot^2\omega - 3}$.

In FIGURE 6 below, triangle $A_1A_2A_3$ is pictured again with the Neuberg circle with centre N. Suppose A_3' is the second point of intersection of A_1A_3 with the Neuberg circle. Then triangles $A_1A_2A_3$ and $A_1A_2A_3'$ are (indirectly) similar. This is because one angle (in this case α_1) and the Brocard angle ω together completely determine the shape of the triangle. As a consequence the lengths of the tangents from A_1 and A_2 to the Neuberg circle are both equal to the length a_3 of the base, or $A_1R_2 = A_1R_1 = A_1A_2 = a_3$. Moreover, $\angle A_2A_1R_1 = \delta_1$ and $\angle A_2A_1R_2 = \delta_2$, the smallest and largest value, respectively, any angle of a triangle with Brocard angle ω can possibly attain. Comparing FIGURES 4 and 6, we observe that running through the Neuberg circle clockwise corresponds to running through the ω-curve of FIGURE 4 counter-clockwise. For instance, the points R_1 and m_1 correspond to the same triangles and so do R_2 and M_1. Finally, what is the locus of the Brocard points Ω and Ω' when A_3 runs

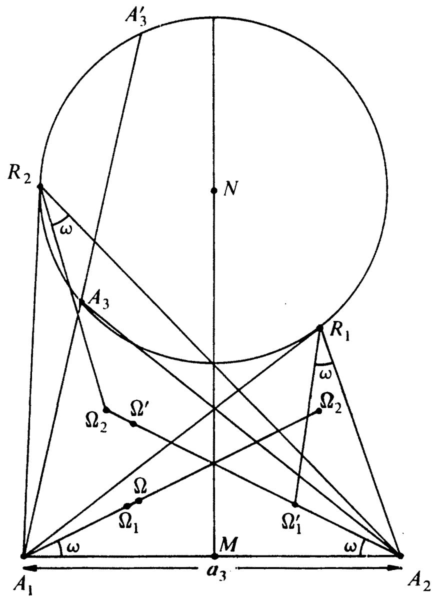

FIGURE 6.
Locus of Ω and Ω' for fixed A_1A_2, Brocard angle ω and orientation.

through the Neuberg circle? Apparently, Ω' runs through the line segment $\Omega_1'\Omega_2'$ twice. Here Ω_i' is the second Brocard point of triangle (T_i), where $(T_1) = A_1A_2R_1$ and $(T_2) = A_1A_2R_2$. A similar line segment $\Omega_1\Omega_2$ is obtained as the locus of Ω by reflection of $\Omega_1'\Omega_2'$ in the line through M and N. It may be verified that both line segments have length

$$\frac{4}{3}a_3\sqrt{1-4\sin^2\omega}\,.$$

The locus of the Brocard point Ω of a triangle with fixed base of length a_3 and Brocard angle ω, as its third vertex A_3 runs through a Neuberg circle, is a line segment of length $\frac{4}{3}a_3\sqrt{1-4\sin^2\omega}$.

A Circulant Matrix

In the previous section we chose a given line segment as the positional fixture for equibrocardal triangles associated with (T), namely, the base of (T). Next we shall fix only one point common to all triangles to be considered. It turns out that the centroid is a good choice. Hence from now on all triangles shall have their centroids coinciding with that of the given triangle (T).

Up to this point, we have used methods and arguments of a geometric and trigonometric nature only. But we shall see that complex numbers and a little linear algebra also prove to be particularly useful.

Let triangle (T) be situated in the complex plane, so that its vertices are given by complex numbers z_1, z_2, and z_3. For obvious reasons we shall use capital Z's instead of capital A's to indicate these vertices. Also we shall change (T) into (Z). Since the centroid of (Z) has an important role to play, we choose it to coincide with the origin O. This means that

$$z_1 + z_2 + z_3 = 0.$$

It would be nice if we could find transformations transforming (Z) into equibrocardal triangles, leaving its centroid fixed and such that triangles of all different shapes with the same Brocard angle and orientation as (Z) appear as images under these transformations. Obvious examples of transformations with these properties are those given by the even permutations of the vertices and by rotations about O over a fixed angle, and also homothetic transformations with centre O share these properties. All of these may be regarded as linear or matrix transformations A, transforming complex vectors $z = (z_1, z_2, z_3)$ in unitary space $\mathbb{C}^3$ into complex vectors $w = (w_1, w_2, w_3) \in \mathbb{C}^3$ by means of the vector relation

$$Az = w.$$

For our purpose it suffices to choose only real matrices A. The complex vector z, usually written as a column instead of a row of complex numbers, is associated with the triangle (Z) and w is associated with the triangle (W). For instance, the transformations permuting the vertices of (Z) without changing its orientation are given by the three permutation matrices

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Note that $P_2 = P_1^2$ and $P_0 = P_1^3$. In general, the orientation of (Z) remains unchanged by the transformation A provided $\det(A) > 0$. Further, because of symmetry, it is reasonable to require that

$$A = P_i^{-1} A P_i,$$

for $i = 0, 1, 2$. This means that two triangles (Z) and (W), corresponding by the linear relation $Az = w$, remain so after the same permutation is applied to their vertices. Thus restricted, the matrix A becomes what is known as a circulant matrix, i.e., a matrix of type (see [1])

$$A = \begin{pmatrix} s & t & r \\ r & s & t \\ t & r & s \end{pmatrix}. \tag{9}$$

Recall that the reason for considering matrices or matrix transformations was to find representatives for all differently shaped triangles with the same Brocard angle as the basic triangle $(Z) = Z_1 Z_2 Z_3$. Thus, if we define

$$F(z) = \left(|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2\right)/\Delta(z), \tag{10}$$

where, not surprisingly $\Delta(z)$ denotes the area of (Z), we would like to find out which conditions have to be imposed on A to guarantee that

$$F(Az) = F(z)$$

for all z. Indeed, it follows from (3) that $F(z) = 4\cot\omega$, provided z is the vector associated with the vertices of (Z). First of all, the transformation $w = Az$ causes the area of (Z) to be multiplied by a factor $\det(A)$. Hence $\Delta(w) = \det(A)\Delta(z)$. Note that for real r, s, and t, because (9) holds,

$$\det(A) = r^3 + s^3 + t^3 - 3rst,$$

and, in particular, $\det(A) > 0$. Secondly, on putting $u_i = z_i - z_{i+1}$, where the indices i are taken modulo 3, we get

$$\begin{aligned} \Delta(w)F(w) &= |su_1 + tu_2 + ru_3|^2 + |ru_1 + su_2 + tu_3|^2 + |tu_1 + ru_2 + su_3|^2 \\ &= \left(r^2 + s^2 + t^2 - rs - st - tr\right)\left(|u_1|^2 + |u_2|^2 + |u_3|^2\right), \end{aligned}$$

because of the relation $u_1 + u_2 + u_3 = 0$. We leave the somewhat tedious calculations to the reader. So

$$\begin{aligned} F(z)/F(w) &= \det(A)/(r^2 + s^2 + t^2 - rs - st - tr) \\ &= \left(r^3 + s^3 + t^3 - 3rst\right)/\left(r^2 + s^2 + t^2 - rs - st - tr\right) \\ &= r + s + t. \end{aligned}$$

We conclude that, if A is given by (9) and if $\det(A) > 0$, then $w = Az$ and z are associated with equibrocardal triangles if and only if $r + s + t = 1$.

Further, as rotations about O and homothetic transformations with centre O do not affect the Brocard angle, we may also, without loss of generality, prescribe a particular position for one of the vertices of the image triangle (W). Let us choose W_1 on the line through Z_1 and Z_2. This choice, being equivalent with $r = 0$, forces W_i to be incident with the line through Z_i and Z_{i+1} for each i. Moreover, these points W_i divide the sides of (Z) into equal ratios (see FIGURE 7).

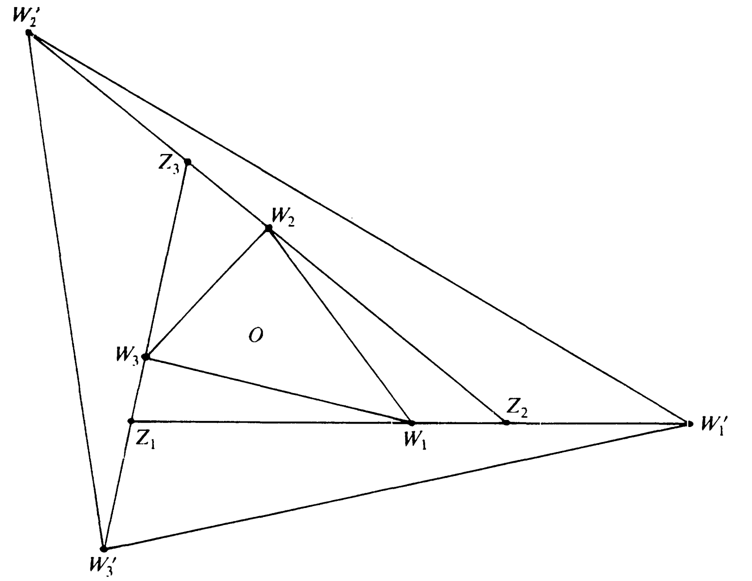

FIGURE 7.
Equibrocardal triangles in the complex plane: $w = Az$, $w' = A'z$; $t = 3/4$ for A, $t = 3/2$ for A'; see (11).

The final form of the circulant A with the required properties now is

$$A = \begin{pmatrix} 1-t & t & 0 \\ 0 & 1-t & t \\ t & 0 & 1-t \end{pmatrix} \quad \text{with } t \in \mathbb{R}. \tag{11}$$

Note that

$$\det(A) = t^3 + (1-t)^3 = t^2 - t(1-t) + (1-t)^2 = 3t^2 - 3t + 1$$

for all choices of $t \in \mathbb{R}$.

So far, we have found a large set of differently shaped equibrocardal triangles as images of a given triangle (Z) under special circulant matrix transformations. But does this set exhaust all possibilities up to orientation and similarity? Surprisingly, the answer is yes, it does. To prove this, we consider a certain triangle transformation σ_λ in the complex plane, which resembles a sort of distorted reflection in the real axis (see FIGURE 8). For each triangle (Z) in the upper half plane H, let $\sigma_\lambda(z) = w$ be associated with the triangle in the lower half-plane, derived from (Z) by multiplying the imaginary parts of z_i by a constant factor $-\lambda$ $(0 < \lambda \leqslant 1)$. In other words, $\mathrm{Re}(w_i) = \mathrm{Re}(z_i)$ and $\mathrm{Im}(w_i) = -\lambda\,\mathrm{Im}(z_i)$ for $i = 1,2,3$. The same effect is obtained by the orthogonal projection in 3-space of a triangle in a given plane onto a second plane. What effect does this transformation σ_λ have on the Brocard angle? To find out, we reconsider the function $F(z)$ of (10). As before, $u_i = z_i - z_{i+1}$. A straightforward calculation shows that

$$2\Delta(w)F(w) = (1+\lambda^2)\Delta(z)F(z) + (1-\lambda^2)\,\mathrm{Re}\big(u_1^2 + u_2^2 + u_3^2\big).$$

Since $\Delta(w) = \lambda\Delta(z)$, we may also write

$$2\lambda F(w)/F(z) = 1 + \lambda^2 + (1-\lambda^2)E(z),$$

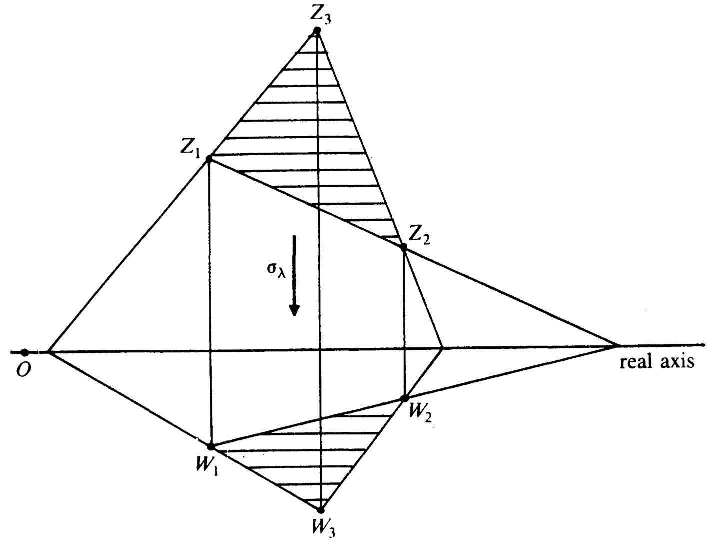

FIGURE 8.
Distorted reflection $\sigma_\lambda(\lambda = 1/2)$: $\text{Re}(w_i) = \text{Re}(z_i), \text{Im}(w_i) = -\lambda \text{Im}(z_i)$.

or

$$2\cot \omega_\lambda / \cot \omega = \lambda^{-1} + \lambda + (\lambda^{-1} - \lambda)E(z),$$

where ω_λ is the Brocard angle of (W) and the expression $E(z)$, defined by

$$E(z) = \text{Re}\{(u_1^2 + u_2^2 + u_3^2)/(|u_1|^2 + |u_2|^2 + |u_3|^2)\},$$

depends on the shape of (Z) only. In case of an equilateral triangle (Z), the expression $E(z)$ vanishes. To prove this, assume that $|u_1| = |u_2| = |u_3|$ and define $v_i = u_i/|u_i|$ for $i = 1,2,3$. Then $E(z)$ satisfies

$$3E(z) = \text{Re}(v_1^2 + v_2^2 + v_3^2).$$

Now $v_1 + v_2 + v_3 = 0$ as $u_1 + u_2 + u_3 = 0$. Further, v_1, v_2, and v_3 lie on the unit circle, which implies that v_2/v_1 and v_3/v_1 are cubic roots of unity with $v_2/v_1 + v_3/v_1 = -1$. Hence $v_2/v_1 = \rho$ and $v_3/v_1 = \rho^2$ so that

$$v_1^2 + v_2^2 + v_3^2 = v_1^2(1 + \rho + \rho^2) = 0$$

as required. Moreover,

$$2\cot \omega_\lambda = (\lambda^{-1} + \lambda)\sqrt{3},$$

as $\cot(\pi/6) = \sqrt{3}$.

This shows that the resulting Brocard angle ω_λ merely depends on the multiplication factor λ! As a consequence, any two equibrocardal triangles (W^1) and (W^2) in the lower half-plane may be seen as the images of two equilateral triangles (Z^1) and (Z^2), respectively, in H by the same transformation σ_λ:

$$\sigma_\lambda(z^i) = w^i, \qquad i = 1,2.$$

Without loss of generality we may assume that the centroids of (W^1) and (W^2)

coincide. Then also (Z^1) and (Z^2) have the same centroid C. Clearly, there is a homothetic transformation (such a transformation multiplies the distance between every two points by the same constant factor) with centre C transforming (Z^2) into $(Z^2)'$ such that the vertices of the latter triangle are incident with the sides of the former, one vertex of $(Z^2)'$ on each side of (Z^1). See FIGURE 9 below.

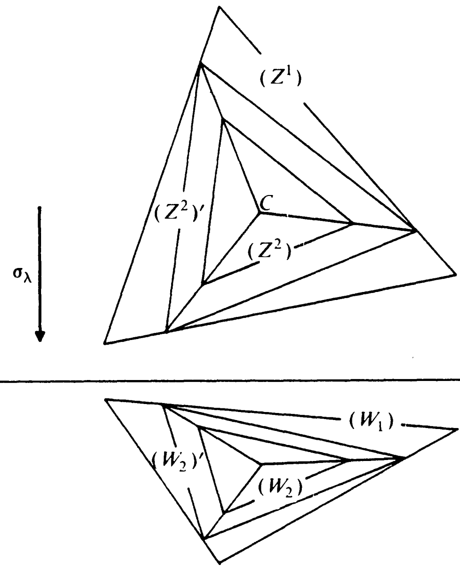

FIGURE 9.
Equilateral triangles are mapped onto equibrocardal triangles by the distorted reflection σ_λ.

Since both (Z^1) and $(Z^2)'$ are equilateral, the vertices of $(Z^2)'$ divide the sides of (Z^1) into equal ratios. The corresponding triangles (W^1) and $(W^2)'$ have the same property, because the transformation σ_λ preserves ratios. This proves that any triangle, agreeing in both orientation and Brocard angle with a given triangle (Z), is directly similar to the triangle onto which (Z) is mapped by a suitable circulant matrix transformation of type (11).

Triangles, the vertices of which divide the sides of a given triangle with Brocard angle ω cyclically in the same ratios, are equibrocardal. Except for similarity and orientation of vertices these triangles exhaust all possible triangles with Brocard angle ω.

Finally, we would like to know in what way the points of the ω-curve of FIGURE 4 correspond to the triangles (W), where $w = Az$, A is a circulant matrix of type (11) and (Z) is the given triangle with Brocard angle ω. Clearly, when t runs through $\mathbb{R}$, the point on the ω-curve corresponding to t, runs through this curve in a counterclockwise fashion; the point indicated by (T) corresponds to $t = 0$. The part of the ω-curve between the points m_2 and m_3—note that it is contained in the shaded area of FIGURE 4—corresponds approximately to the t-interval $-0.23 \leqslant t \leqslant 0.46$.

Descartes' Folium

This final section is devoted to a description of the locus of the Brocard points as their associated triangles move through the Brocard configuration discussed in the previous section. Because of the beauty of the final result, going through the (sometimes tedious) derivations is certainly worth the trouble.

In FIGURE 10 we have gathered the necessary information obtained in the foregoing sections. For convenience we take the positive Brocard point Ω of (Z) to be the origin O of the complex plane. The complex number associated with the positive Brocard point of triangle (W) shall be denoted by ω_t, so that $\omega_0 = 0$.

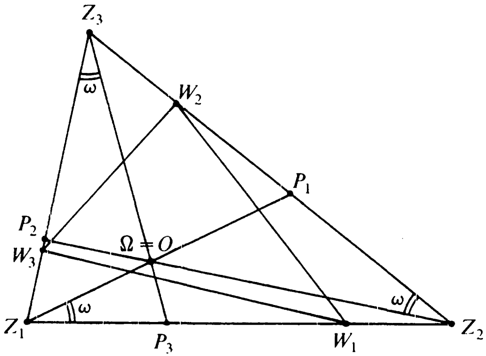

FIGURE 10.
$|z_i - w_i| = t|z_i - z_{i+1}|$.

It is obvious that every complex number α can be written in one way only as a 'convex' combination of z_1, z_2 and z_3, to be precise, $\alpha = c_1z_1 + c_2z_2 + c_3z_3$ for a unique triple of real numbers c_1, c_2, and c_3 with $c_1 + c_2 + c_3 = 1$. In particular

$$0 = \omega_0 = \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \tag{12}$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Since Z_3, Ω, and P_3 are collinear (see FIGURE 10), there is a real number c such that

$$\omega_0 = (1-c)p_3 + cz_3.$$

Naturally, p_3 is the complex number associated with the point P_3. Also

$$p_3 = c_1z_1 + c_2z_2$$

with $c_1 + c_2 = 1$, because Z_1, Z_2, and P_3 are collinear. Consequently,

$$\omega_0 = (1-c)c_1z_1 + (1-c)c_2z_2 + cz_3.$$

Comparing this with (12), we may deduce that $c = \lambda_3$, because of the uniqueness of this expression. The number c has an obvious interpretation, namely, as the ratio of the directed line segments $P_3\Omega$ and P_3Z_3.

To calculate c and hence λ_3, we observe that the triangles $Z_1\Omega P_3$ and $Z_3Z_1P_3$ are similar, which implies

$$A_1P_3/A_3P_3 = P_3\Omega/P_3A_1 \quad \text{or} \quad c = (A_1P_3/A_3P_3)^2.$$

The latter expression may be written, by the rule of sines in triangle $Z_1P_3Z_3$, as

$$(\sin\omega/\sin\alpha_1)^2 = (\sin\omega/\Delta(z))^2|(z_1-z_2)(z_3-z_1)|^2/4.$$

Hence (12) can be rewritten as

$$(2\Delta(z)/\sin\omega)^2\omega_0 = |u_1u_2|^2z_1 + |u_2u_3|^2z_2 + |u_3u_1|^2z_3, \tag{13}$$

because of symmetry. Recall that $u_i = z_i - z_{i+1}$ and that $\Delta(z)$ signifies the area of triangle (Z). The corresponding formula for (W) may be derived analogously. Before giving this formula explicitly, let us renew the habit of taking indices modulo 3. Let us also agree to the following abbreviated notation:

$$\Sigma_i e_i = e_1 + e_2 + e_3,$$

where the sum Σ_i extends over $i = 1,2,3$. Hence, the right-hand side of (13) in abbreviated form looks like

$$\Sigma_i |u_i u_{i+1}|^2 z_i.$$

The promised formula for (W) now may be written as

$$(2\Delta(w)/\sin\omega)^2 \omega_t = \Sigma_i |(w_i - w_{i+1})(w_{i+1} - w_{i+2})|^2 w_i. \tag{14}$$

As we want to make explicit the dependence of (14) on the parameter t, we substitute

$$w_i = (1-t)z_i + tz_{i+1}.$$

Clearly, $|w_i - w_{i+1}|^2$ is a quadratic polynomial in t. This allows us to define

$$p_i(t) = |w_i - w_{i+1}|^2 = a_i t^2 + b_i t(1-t) + c_i(1-t)^2. \tag{15}$$

A few simple properties of the coefficients of $p_i(t)$ are readily established. For instance

$$a_i = c_{i+1} = |u_{i+1}|^2 \quad \text{and} \quad a_i + b_i + c_i = a_{i+1}. \tag{16}$$

The former is the result of the substitutions $t = 1$ and $t = 0$, and the latter follows from the substitution $t = 1/2$ in conjunction with $\Sigma_i u_i = 0$. Also

$$(2\Delta(z)/\sin\omega)^2 = 4\Delta^2(z)\Sigma_i(1/\sin^2\alpha_{i+1}) = \Sigma_i a_i c_i,$$

because of (2) and the observation that for each i

$$4\Delta^2(z) = a_i c_i \sin^2\alpha_{i+1}.$$

All these notational simplifications are intended to give (13) and (14) a less complicated appearance. Thus (13) becomes

$$0 = \omega_0 \Sigma_i a_i c_i = \Sigma_i a_i c_i z_i \tag{17}$$

and (14) eventually looks like

$$\{\Delta(w)/\Delta(z)\}^2 \omega_t \Sigma_i a_i c_i = \Sigma p_i(t) p_{i+1}(t)(1-t)z_i + tz_{i+1} \tag{18}$$

or

$$\{\Delta(w)/\Delta(z)\}^2 \omega_t \Sigma_i a_i c_i = \Sigma_i \{(1-t)p_i(t)p_{i+1}(t) + tp_i(t)p_{i+2}(t)\} z_i.$$

Now the left-hand side of (18) is the product of ω_t and a quartic polynomial in t, because

$$\{\Delta(w)/\Delta(z)\}^2 = \{\det(A)\}^2 = (3t^2 - 3t + 1)^2. \tag{19}$$

Concentrating on the right-hand side of (18), which we shall denote by $P(t)$, we see that it is a polynomial of degree 5 in t with complex coefficients. Obviously, $P(0)=P(1)=0$ because of (17). Hence, as a polynomial in $\mathbb{C}[t]$, $P(t)$ is divisible by $t(1-t)$. It can also be shown that $P(t)$ is divisible by the polynomial $\det(A)=3t^2-3t+1$. In fact,

$$P(t)=t(1-t)(3t^2-3t+1)\{\alpha t+\beta(1-t)\}\Sigma_i a_i c_i, \tag{20}$$

where the complex numbers α and β are defined by

$$\alpha\Sigma_i a_i c_i=\Sigma_i(a_i^2+b_i c_i)z_i, \qquad \beta\Sigma_i a_i c_i=\Sigma_i(c_i^2+a_i b_i)z_i.$$

In establishing this result, frequent use is made of (17).

Inserting our findings (19) and (20) into (18) yields

$$\begin{aligned}\omega_t&=\{t^2(1-t)\alpha+t(1-t)^2\beta\}/(3t^2-3t+1)\\&=(\tau^2\alpha+\tau\beta)/(1+\tau^3),\end{aligned}$$

where $\tau=t/(1-t)$. We are nearly through, because on putting

$$X=\tau^2/(1+\tau^3), \qquad Y=\tau/(1+\tau^3),$$

so that

$$\omega_t=X\alpha+Y\beta,$$

and letting t run through all real values $\neq 1$, an unexpected curve emerges as the locus of Ω_t, namely the curve given by

$$X^3+Y^3=XY, \tag{21}$$

in the (X,Y)-coordinate system with basis $\{\alpha,\beta\}$. We recognize this curve, usually given by (21) relative to an orthogonal coordinate system, as the famous Folium of Descartes. Information on this cubic curve can be found in most classical texts on analytic geometry or on plane algebraic curves.

Note that

$$\alpha+\beta=2\omega_{1/2}=\Sigma_i z_i,$$

which shows that the centroid lies on the line segment $\Omega\Omega_{1/2}$ at two-thirds of its length as seen from $\Omega=0$. Also the points on the closed loop of the curve correspond to the t-values between 0 and 1. In FIGURE 11 below we see the original triangle (Z) and the corresponding locus of Ω_t.

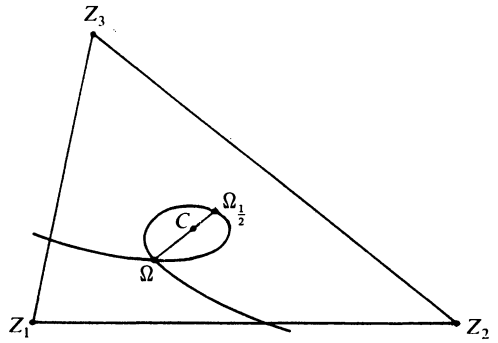

FIGURE 11.
The locus of the Brocard point Ω_t of (W) as in FIGURE 10.

If t runs through $\mathbb{R}$, each of the Brocard points of the triangle with vertices given by $w = Az$, where A has the form (11) *and z corresponds to the vertices of the given triangle, runs through a twisted version of the Folium of Descartes.*

The author wishes to express his thanks to the referees for their helpful comments, in particular for suggesting a better title.

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A Nonconstructible Isomorphism

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Very often we need to make a careful distinction between those mathematical objects which merely exist and those objects which are actually constructible. Metaphysically speaking, "God" is, of course, one such object. But it is rather difficult to find simple and concrete examples of such objects suitable for a relatively lower (say, at an advanced undergraduate) level. In this note, I would like to provide one such down-to-earth example of an isomorphism which can be explained to any class that has had a dose of elementary linear algebra and calculus. Apart from being a nondescriptive isomorphism, this example also demonstrates (i) the importance of dimension, (ii) the so-called cardinality arguments, (iii) the intricacy of primes, rationals, and irrationals, and finally (iv) the use of 'external' machinery (here vector spaces, to prove a result within group theory) to dig this truth from the "deep well" of mathematics (see J. Larmor [1]).

FACT 1: There is an isomorphism from the additive group $\mathbb{R} = \langle R; + \rangle$ of all reals to the multiplicative group $\mathbb{R}^* = \langle R^+; \cdot \rangle$ of all positive reals.

Proof. The exponential map $x \to e^x$ is an isomorphism because $e^{x+y} = e^x \cdot e^y$ and e^x is one-to-one, onto, and always positive.

FACT 2: There is no isomorphism from the additive group $\mathbb{Q} = \langle Q; + \rangle$ of all rationals to the multiplicative group $\mathbb{Q}^* = \langle Q^+; \cdot \rangle$ of all positive rationals.

Proof. In the additive group $\mathbb{Q}$, given any element a, the group equation $x + x = a$ can be solved for x, namely $x = a/2$ ($\mathbb{Q}$ is after all a divisible group), but as we know, the corresponding equation $x \cdot x = a$ cannot be solved among the rationals for all a in Q^+ (for example, $\sqrt{2}$ is irrational).

Now, between Q and R lies the nice class A of all algebraic real numbers (i.e., the set of all real numbers which are solutions of polynomial equations with integer coefficients). Unlike Q, both the group structures based on A and A^+ are divisible; but at the same time the exponential of an algebraic number is always transcendental (except in the obvious trivial case $e^0 = 1$; see W. J. LeVeque [2], Corollary 1, page 186) and thus the techniques of both of the proofs above fail miserably for algebraic numbers. In spite of these deviations, we have the following:

FACT 3: There is an isomorphism from the additive group $\mathbb{A} = \langle A; + \rangle$ of all algebraic numbers to the multiplicative group $\mathbb{A}^* = \langle A^+; \cdot \rangle$ of all positive algebraic numbers.

Proof. It is clear that the additive group $\mathbb{A}$ of all algebraic numbers can be thought of as a vector space over the field Q of all rationals because if a is algebraic and q is rational then qa is algebraic and so we have a scalar multiplication. Also, if a is algebraic and q is a rational number, then a^q is algebraic. Using this, one can easily convert the abelian group $\mathbb{A}^*$ into a vector space over Q by simply defining the scalar multiplication as $q \circ a = a^q$. The laws of indices learned in high school show that this satisfies the properties of scalar multiplication:

$$q \circ (a \cdot b) = (ab)^q = a^q b^q = (q \circ a) \cdot (q \circ b),$$
$$(q + r) \circ a = a^{q+r} = a^q a^r = (q \circ a) \cdot (r \circ a),$$
$$q \circ (r \circ a) = q \circ (a^r) = (a^r)^q = a^{qr} = (qr) \circ a, \quad \text{and} \quad 1 \circ a = a^1 = a$$

It is obvious that no finite number of algebraic numbers can generate all algebraic numbers in either of the vector spaces (the various roots of prime numbers are algebraically independent) and both the sets A and A^+ are of the same cardinality and hence they both are vector spaces over Q and are of the same (countably infinite) dimension. But given a field Q and a cardinal number n, there exists a *unique* vector space of dimension n over Q. Thus as vector spaces, and hence *a posteriori* as abelian groups, these two structures are, indeed, isomorphic.

This only means that *there exists* an isomorphism between these two groups. But, unlike the case of the real numbers, it is difficult actually to exhibit such an isomorphism. Among other things, such an isomorphism cannot preserve continuity or order because any such order preserving or continuous function will automatically be an exponential function. To obtain such an isomorphism, we need what is known as a Hamel basis for the vector spaces in question which, in turn, depends upon transfinite induction. For a lively discussion on this topic, see Section 3 of Chapter V in [4]. The reader is also invited to read the entertaining article "Constructive Mathematics" by Mark Mandelkern which appeared in a recent issue of this MAGAZINE (see [3]).

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Some Algorithms for the Sums of Integer Powers

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1. Let $S_k(n) = 1^k + 2^k + \cdots + n^k$, where k, n are positive integers. The usual method of evaluation of $S_k(n)$ for $k = 1, 2, \ldots$ is the identity

$$\sum_{i=0}^{k} \binom{k+1}{i} S_i(n) = (n+1)^k - 1. \tag{1}$$

As usual, $\binom{k}{i}$ denotes the binomial coefficient $k!/i!(k-i)!$.

C. Kelly [1] presents a very elementary proof for (1) using the binomial expansion. A slightly different proof follows. We have the identity

$$\sum_{j=1}^{n} \left[(j+1)^{k+1} - j^{k+1} \right] = (n+1)^{k+1} - 1, \tag{2}$$

which can be written in the form

$$\sum_{j=1}^{n} \left[\binom{k+1}{k} j^k + \binom{k+1}{k-1} j^{k-1} + \cdots + \binom{k+1}{1} j + 1 \right] = (n+1)^{k+1} - 1.$$

Hence it follows that

$$\binom{k+1}{k} \sum_{j=1}^{n} j^k + \binom{k+1}{k-1} \sum_{j=1}^{n} j^{k-1} + \cdots + \binom{k+1}{1} \sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 = (n+1)^{k+1} - 1,$$

which yields (1).

Using identities of type (2) we can find formulas of type (1) involving only odd subscripts i or only even subscripts i.

2. Thus, let us consider the identity

$$\sum_{j=1}^{n} \left[(j+1)^{k+1} - (j-1)^{k+1} \right] = (n+1)^{k+1} + n^{k+1} - 1.$$

By virtue of the binomial expansion, we deduce that

$$\binom{k+1}{1} S_k(n) + \binom{k+1}{3} S_{k-2}(n) + \binom{k+1}{5} S_{k-4}(n) + \cdots$$
$$= \frac{(n+1)^{k+1} + n^{k+1} - 1}{2}. \tag{3}$$

If in (3) we take $k = 2p,\ p = 0, 1, \ldots$, then we get

$$\binom{2p+1}{1}S_{2p}(n)+\binom{2p+1}{3}S_{2p-2}(n)+\cdots+\binom{2p+1}{2p+1}S_0(n)$$

$$=\frac{(n+1)^{2p+1}+n^{2p+1}-1}{2}. \tag{4}$$

Now if in (3) we take $k=2p-1$, $p=1,2,3,\ldots$ we reach

$$\binom{2p}{1}S_{2p+1}(n)+\binom{2p}{3}S_{2p-3}(n)+\cdots+\binom{2p}{2p-1}S_1(n)=\frac{(n+1)^{2p}+n^{2p}-1}{2}. \tag{5}$$

In particular, from (4) it follows:

$p=0,\quad S_0(n)=n$

$p=1,\quad 3S_2(n)+S_0(n)=\dfrac{(n+1)^3+n^3-1}{2},\quad$ whence $\quad S_2(n)=\dfrac{n(n+1)(2n+1)}{6}$

$p=2,\quad 5S_4(n)+10S_2(n)+S_0(n)=\dfrac{(n+1)^5+n^5-1}{2},\quad$ whence

$$S_4(n)=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Similar formulas for S_{2p+1} can be derived from (5).

3. For the sums $S_1(n)$, $S_3(n),\ldots,$ using the identity

$$\sum_{j=1}^{n}\left[j^k(j+1)^k-j^k(j-1)^k\right]=n^k(n+1)^k,$$

we deduce

$$\binom{k}{1}S_{2k-1}(n)+\binom{k}{3}S_{2k-3}(n)+\cdots=\frac{n^k(n+1)^k}{2}, \tag{6}$$

$k=1,2,3,\ldots.$

From (6), we get:

$k=1,\quad S_1(n)=\dfrac{n(n+1)}{2}$

$k=2,\quad 2S_3(n)=\dfrac{n^2(n+1)^2}{2},\quad$ whence $\quad S_3(n)=\dfrac{n^2(n+1)^2}{4}$

$k=3,\quad 3S_5(n)+S_3(n)=\dfrac{n^3(n+1)^3}{2},\quad$ whence $\quad S_5(n)=\dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$

$k=4,\quad 4S_7(n)+4S_5(n)=\dfrac{n^4(n+1)^4}{2},\quad$ whence

$$S_7(n)=\frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24}.$$

We notice that the formulas (3), (4), (5), (6) are to be found in [3], where it is proved that they follow from a theorem of Nielson [2].

Remark. Proceeding as above, we can obtain identities of the types (1), (3), (4), (5), (6) for the sums

$$s_k(n) = a_1^k + a_2^k + \cdots + a_n^k, \qquad k = 0, 1, 2, \ldots,$$

where $a_1, a_2, \ldots, a_n, \ldots$ is an arithmetic progression.

Acknowledgement. The author wishes to thank both the referees for their useful suggestions.

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The Factorization of $x^5 \pm x + n$

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It is a surprising fact that $x^5 - x + 2759640$ factors as the product

$$(x^2 + 12x + 377) \times (x^3 - 12x^2 - 233x + 7320).$$

In fact, the quintic,

$$x^5 \pm x + n, \tag{1}$$

rarely factors at all. It is the purpose of this note to find all n for which (1) is reducible.

Clearly, (1) has the linear factor $x + a$ if and only if n is of the form $a^5 \pm a$. So we are more interested in the question of when does (1) factor as the product of a quadratic polynomial and a cubic polynomial.

Assuming that we have the factorization

$$x^5 + mx + n = (x^2 + ax + b)(x^3 + cx^2 + dx + e),$$

we can equate like powers of x in succession to find:

$$c = -a$$
$$d = a^2 - b$$
$$e = a(2b - a^2)$$
$$m = 3a^2b - a^4 - b^2 \tag{2}$$

and

$$n = ab(2b - a^2). \tag{3}$$

Eliminating b from (2) and (3) yields

$$n^2 + (4am - 11a^5)n + a^2(m + a^4)(4m - a^4) = 0.$$

This is a quadratic in n whose solution is

$$n = \frac{11a^5 - 4am \pm 5a^3\sqrt{5a^4 - 4m}}{2}. \tag{4}$$

In order for n to be integral, we must have $5a^4 - 4m = z^2$ for some integral z. Since we are interested in the cases where $m = \pm 1$, we must solve the Diophantine equation $z^2 - 5a^4 = \pm 4$. Let $x = a^2$. Note that x and z must have the same parity, so that we may let $y = (x + z)/2$, where y is also an integer. This puts the equation in the form

$$(2y - x)^2 - 5x^2 = \pm 4$$

or, equivalently,

$$y^2 - xy - x^2 = \pm 1, \tag{5}$$

where it is desired that x be a perfect square.

Equation (5) brings to mind a known property of Fibonacci numbers, namely, that the integer x is a Fibonacci number if and only if there is an integer y such that $y^2 - xy - x^2 = \pm 1$. (This is proven in [3] and [4].) Thus we see that $x = a^2$ must be a Fibonacci number.

But it is also known that the only square Fibonacci numbers are 0, 1, and 144 (see [1] or [2]). If $a^2 = 0$, then $n = 0$ and several trivial factorizations are possible. These will be excluded from the following discussion. Let us now consider the two cases, $m = +1$ and $m = -1$.

Case 1. $m = +1$.

If $a^2 = 1$, then $a = \pm 1$ and using (4) to find n gives $n = \pm 1$ or $n = \pm 6$. If $a^2 = 144$ then $a = \pm 12$, but the values of n obtained do not make $5a^4 - 4$ a perfect square so are ruled out.

Case 2. $m = -1$.

If $a^2 = 1$, then $a = \pm 1$ and $n = 0$ or $n = \pm 15$. If $a^2 = 144$, then $a = \pm 12$ and $n = \pm 22{,}440$ or $n = \pm 2{,}759{,}640$.

We can summarize our results by the following theorems:

THEOREM 1. *The only integral n for which $x^5 + x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 1$ and $n = \pm 6$. The factorizations are*

$$x^5 + x \pm 1 = (x^2 \pm x + 1)(x^3 \mp x^2 \pm 1)$$
$$x^5 + x \pm 6 = (x^2 \pm x + 2)(x^3 \mp x^2 - x \pm 3).$$

THEOREM 2. *The only integral n for which $x^5 - x + n$ factors into the product of an irreducible quadratic and an irreducible cubic are $n = \pm 15$, $n = \pm 22{,}440$, and $n = \pm 2{,}759{,}640$. The factorizations are*

$$x^5 - x \pm 15 = (x^2 \pm x + 3)(x^3 \mp x^2 - 2x \pm 5)$$
$$x^5 - x \pm 22440 = (x^2 \mp 12x + 55)(x^3 \pm 12x^2 + 89x \pm 408)$$
$$x^5 - x \pm 2759640 = (x^2 \pm 12x + 377)(x^3 \mp 12x^2 - 233x \pm 7320).$$

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Symmetry in Probability Distributions

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If one suspects that the probability of occurrence of a certain event is $1/2$, one should also suspect, even though there need not be any, that there will be a simple symmetry proof for the result. One such problem was treated in a previous note [1] where we considered some ramifications of the problem of determining the probability $P(m, n)$ that A gets more heads than B if A and B toss $n+m$ and n fair coins, respectively. As usual, it is assumed that the tosses of each coin are independent. The problem for $P(1, n)$ appears in [2] where Uspensky solves the problem in two ways. In the first way he obtains the double sum

$$P(1,n)=2^{-2n-1}\sum_{r=0}^{n}\sum_{s=r+1}^{n+1}\binom{n}{r}\binom{n+1}{s}, \tag{1}$$

which he sums by means of two combinatorial identities. His second method is much more elegant. Letting A_1 and B_1 denote the number of heads tossed by A and B, respectively, and A_2 and B_2 the respective number of tails, it follows that

$$\operatorname{Prob}\{A_1>B_1\}=\operatorname{Prob}\{A_2>B_2\}$$

since the coins are fair. Now $A_2>B_2$ is the same as $n+1-A_1>n-B_1$ or $A_1\leqslant B_1$. Since $A_1>B_1$ and $A_1\leqslant B_1$ exhaust all the possible outcomes,

$$\operatorname{Prob}\{A_1>B_1\}=\operatorname{Prob}\{A_1\leqslant B_1\}=1/2.$$

This same elegant method applies to the more general case where A and B have arbitrary but symmetric distributions. If $A(r)$ and $B(r)$ denote the probabilities that A and B get r heads in a toss of $n+m$ and n coins, respectively, then $A(r)=A(n+m-r)$, $B(r)=B(n-r)$, and

$$P(m,n)=\sum_{r=0}^{n}\sum_{s=r+1}^{n+m}B(r)A(s). \tag{2}$$

For the special case $m=1$, we again have $P(1,n)=1/2$ where here

$$P(1,n)=\sum_{r=0}^{n}\sum_{s=r+1}^{n+1}B(r)A(s). \tag{3}$$

Although (1) was summed by using two combinatorial identities, it could be summed in a simpler fashion by employing the symmetry of the two distributions. We show this for the sum in (3). Let $s=n+1-s'$, and $r=n-r'$ to give (dropping the primes)

$$P(1,n)=\sum_{r=0}^{n}\sum_{s=0}^{r}B(r)A(s). \tag{4}$$

Then adding (3) and (4), we get

$$2P(1,n)=\sum_{r=0}^{n}\sum_{s=0}^{n+1}B(r)A(s). \tag{5}$$

Since the latter sum includes all the possible outcomes it equals 1 and thus $P(1,n)=1/2$. Also, in a similar manner we can show that

$$2P(0,n)=1-\sum_{r=0}^{n}B(r)A(r), \tag{6}$$

$$2P(2,n)=1+\sum_{r=0}^{n}B(r)A(r+1). \tag{7}$$

We now consider analogous results for continuous distributions. If A and B have the same continuous density function $\rho(x)$ for $-\infty<x<\infty$, it is a known elementary combinatorial result from order statistics [3] that $\text{Prob}\{A\geqslant B\}=1/2$ (again it is assumed here and subsequently that A and B are independent). The same result holds if A and B have different continuous but symmetric density functions $A(x)$, $B(x)$, i.e., $A(x)=A(-x)$, $B(x)=B(-x)$ for $-\infty<x<\infty$. A direct proof, analogous to the one for (3), follows by letting $x=-x'$ and $y=-y'$ in

$$\text{Prob}\{A\geqslant B\}=\int_{-\infty}^{\infty}B(y)\,dy\int_{y}^{\infty}A(x)\,dx$$

to give (dropping the primes)

$$\text{Prob}\{A\geqslant B\}=\int_{-\infty}^{\infty}B(y)\,dy\int_{-\infty}^{y}A(x)\,dx.$$

Then by adding,

$$2\,\text{Prob}\{A\geqslant B\}=\int_{-\infty}^{\infty}B(y)\,dy\int_{-\infty}^{\infty}A(x)\,dx=1.$$

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PROBLEMS

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Proposals

To be considered for publication, solutions should be received by November 1, 1988.

1297. *Proposed by Irl C. Bivens and Benjamin G. Klein, Davidson College, North Carolina.*

For k a positive integer, define A_n for $n = 1, 2, \dots$ by

$$A_{n+1} = \frac{nA_n + 2(n+1)^{2k}}{n+2}, \qquad A_1 = 1.$$

Prove that A_n is an integer for all $n \geqslant 1$, and A_n is odd if and only if n is congruent to 1 or 2 modulo 4.

1298. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

A quadrilateral $ABCD$ is circumscribed about a circle, and P, Q, R, S are the points of tangency of sides AB, BC, CD, DA respectively. Let $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$, and $p = |QS|$, $q = |PR|$. Show that

$$\frac{ac}{p^2} = \frac{bd}{q^2}.$$

1299. *Proposed by Isaac J. Schoenberg, Madison, Wisconsin.*

Eight positive point-weights of masses m_i $(i = 0, 1, \dots, 7)$ are placed, respectively, on the vertices A_i $(i = 0, 1, \dots, 7)$ of a regular octagon, and they are such that their centroid is at the center of the octagon. Assume all eight numbers m_i are *rational* numbers. Show that they balance out in diametrically opposite pairs; i.e., $m_0 = m_4$, $m_1 = m_5$, $m_2 = m_6$, $m_3 = m_7$.

ASSISTANT EDITORS: CLIFTON CORZATT and THEODORE VESSEY, *St. Olaf College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren C. Larson, Department of Mathematics, St. Olaf College, Northfield, MN 55057.

1300. *Proposed by Edward Kitchen, Santa Monica, California.*

Let r, s, n be positive integers such that $r+s=n$. Prove that

$$\{1,2,\dots,n-2\}=\left\{\left\lfloor\frac{n}{r}\right\rfloor,\left\lfloor\frac{2n}{r}\right\rfloor,\dots,\left\lfloor\frac{(r-1)n}{r}\right\rfloor\right\}\cup\left\{\left\lfloor\frac{n}{s}\right\rfloor,\left\lfloor\frac{2n}{s}\right\rfloor,\dots,\left\lfloor\frac{(s-1)n}{s}\right\rfloor\right\}$$

if and only if both r and s are relatively prime to n.

1301. *Proposed by Vidhyanath K. Rao, Ohio State University at Newark, Newark, Ohio.*

Let T be the ring of symmetric polynomials in $\mathbf{Q}[x,y]$, where $\mathbf{Q}$ denotes the ring of rational numbers. Prove that the *subring* of T generated by $\{a(x^n+y^n): a\in Q, n \text{ odd}\}$ is equal to the *ideal* of T generated by $x+y$.

Quickies

Answers to the Quickies are on page 203.

Q734. *Proposed by Murray S. Klamkin, University of Alberta, Canada.*

Determine the locus of all points whose parametric representation is given by

$$x=\frac{\xi(h\xi+k\eta+l\zeta)}{(\xi^2+\eta^2+\zeta^2)},$$

$$y=\frac{\eta(h\xi+k\eta+l\zeta)}{(\xi^2+\eta^2+\zeta^2)},$$

$$z=\frac{\zeta(h\xi+k\eta+l\zeta)}{(\xi^2+\eta^2+\zeta^2)},$$

where the parameters ξ,η,ζ take on all values in $[0,1]$ and h,k,l are positive constants.

Q735. *Proposed by Norman Schaumberger, Bronx Community College, New York.*

If a, b, and c are not all equal, show that

$$\frac{a+b+c}{3}=\sqrt[3]{abc}$$

if and only if $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=0$.

Solutions

A duality result on subsets **June 1987**

1267. *Proposed by Ronald Graham, AT&T Labs, Murray Hill, New Jersey.*

Let $X=\{1,2,\dots,n\}$ and let S be any nonempty collection of subsets of X. Define S' to be the collection of all subsets of X that are subsets of an odd number of elements of S. Prove that $(S')'=S$.

I. *Solution by James Propp, University of Maryland, College Park.*

Let $\mathscr{P}(X)$ denote the collection of all subsets of X; each set $S \subseteq \mathscr{P}(X)$ is associated with the indicator function $f\colon \mathscr{P}(X) \to \{0,1\}$ satisfying

$$f(A) = \begin{cases} 1 & \text{if } A \in S, \\ 0 & \text{otherwise} \end{cases}$$

for all $A \subseteq X$. Then the collection S' defined in the problem corresponds to the function f', where

$$\begin{aligned} f'(A) &= \begin{cases} 1 & \text{if } \#\{B \supseteq A\colon B \in S\} \text{ is odd}, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \#\{B \supseteq A\colon f(B) = 1\} \text{ is odd}, \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{B \supseteq A} f(B). \end{aligned}$$

(Here, as hereafter, summation is to be interpreted modulo 2.) Proving $(S')' = S$ is equivalent to proving $(f')' = f$. We have

$$\begin{aligned} (f')'(A) &= \sum_{B \supseteq A} f'(B) \\ &= \sum_{B \supseteq A} \sum_{C \supseteq B} f(C). \end{aligned}$$

But the number of occurrences of each particular $f(C)$ in this double sum is

$$\#\{B\colon C \supseteq B \supseteq A\},$$

which is equal to $2^{\#(C) - \#(A)}$ when $C \supseteq A$ and is 0 otherwise; hence the number of times $f(C)$ contributes to $(f')'(A)$ is odd only when $C = A$. Therefore,

$$(f')'(A) = f(A)$$

for all A, so that $(f')' = f$, as claimed.

II. *Solution by Craig Bailey and R. Bruce Richter, U.S. Naval Academy, Annapolis.*

To show that $(S')' = S$, for $S \subseteq \mathscr{P}(X)$, we use the following two lemmas. (Here, $\mathscr{P}(X)$ is the power set of X, and $A \triangle B$ denotes the symmetric difference of A and B.)

LEMMA 1. *Let* $A \subseteq X$. *Then* (a) $\{A\}' = \mathscr{P}(A)$, *and* (b) $(\mathscr{P}(A))' = \{A\}$.

LEMMA 2. $(Q \triangle R)' = Q' \triangle R'$.

To show $(S')' = S$, let $S = \{A_1, \dots, A_n\}$. Then $S = \{A_1\} \triangle \{A_2\} \triangle \cdots \triangle \{A_n\}$. By Lemma 2 and induction, $S' = \{A_1\}' \triangle \{A_2\}' \triangle \cdots \triangle \{A_n\}'$. From Lemma 1(a), $S' = \mathscr{P}(A_1) \triangle \mathscr{P}(A_2) \triangle \cdots \triangle \mathscr{P}(A_n)$. From Lemma 2 and Lemma 1(b), we see that $(S')' = \{A_1\} \triangle \{A_2\} \triangle \cdots \triangle \{A_n\} = \{A_1, A_2, \dots, A_n\} = S$.

Proof of Lemma 1. (a) Evidently, $C \in \{A\}'$ if and only if $C \subseteq A$. (b) If $C \subseteq A$, then C is contained in 2^r subsets of A, where $r = |A - C|$. Therefore, $(\mathscr{P}(A))' = \{A\}$.

Proof of Lemma 2. $C \in (Q \triangle R)'$ if and only if C is a subset of an odd number of elements of $Q \triangle R$, if and only if C is a subset of an odd number of elements either of Q or of R, but not both, if and only if $C \in Q' \triangle R'$.

III. *Solution by Y. H. Harris Kwong, SUNY College at Fredonia, New York.*

Let the $N = 2^n$ subsets of X be $S_1, \dots, S_N$. Define the $N \times N$ matrix $A = (a_{i,j})$ by $a_{i,j} = 1$ if $S_i \subseteq S_j$ and 0 otherwise. Given any collection S of subsets of X, define $C(S)$ to be the column vector $(c_1, \dots, c_N)^t$, where $c_i = 1$ if $S_i \in S$ and 0 otherwise. Let $AC(S) = (x_1, \dots, x_N)^t$. Then x_i is exactly the number of elements of S that contain S_i as a subset. Thus, $AC(S) \equiv C(S') \pmod 2$. To prove that $(S')' = S$, it suffices to show that $A^2 \equiv (b_{i,j}) \pmod 2$ is the identity matrix.

For any i, j and k, $a_{i,k}a_{k,j} = 1$ if and only if $a_{i,k} = a_{k,j} = 1$, which is true if and only if $S_i \subseteq S_k \subseteq S_j$. Thus, $b_{i,j} = \sum_{k=1}^{N} a_{i,k}a_{k,j}$ counts the number of S_k such that $S_i \subseteq S_k \subseteq S_j$. Therefore,

$$b_{i,j} = \begin{cases} 0 & \text{if } S_i \not\subseteq S_j, \\ 2^{|S_j| - |S_i|} & \text{if } S_i \subseteq S_j. \end{cases}$$

Hence, $b_{i,j} \equiv \delta_{i,j} \pmod 2$, or equivalently, $A^2 \equiv I \pmod 2$.

Also solved by Mangho Ahuja, Lee Badger, S. F. Barger, Irl C. Bivens and Stephen L. Davis, Sul-young Choi, George Day, Richard A. Gibbs, T. Jager, Chandrashekhar Khare (student, England), William L. Maier, Fouad Nakhli (student), James Propp (second solution), Matt Richey, Martin Santavy (Canada), Zun Shan and E. T. H. Wang (Canada), Robert W. Sheets, Jan Söderkvist (Sweden), Daniel Ullman, and the Western Maryland College Problems Group.

Fred Galvin (University of Kansas), and *Rodica Simion* and *Frank Schmidt* (Bryn Mawr College), noted that the problem is a dualized version of Problem 5856, *American Mathematical Monthly*, October 1973, pp. 950–951. The two problems are related by $\mathcal{F}' = \mathcal{F}^{c*c}$, where $\mathcal{F}$ is a nonempty family of subsets, $\mathcal{F}^c$ is the family of corresponding complements, and $\mathcal{F}'$, $\mathcal{F}^*$ are as in the respective problems.

Expected waiting time **June 1987**

1268. *Proposed by Lawrence Stout, Illinois Wesleyan University, Bloomington.*

An experiment with probability p of success is repeated. Each time a failure occurs, \$1 is put into a kitty. Each time a success occurs, the contents of the kitty are paid out as a jackpot.

a. What is the expected waiting time until the kitty reaches $\$n$ for the first time starting from an empty kitty?

b. What is the expected waiting time until a jackpot of exactly $\$n$ is won?

Solution by Bruce R. Johnson, University of Victoria, Canada.

We use the powerful tool of conditioning not only to solve parts (a) and (b), but also to find the expected waiting time until a jackpot of at least $\$n$ is won. We begin by defining three random variables:

X = number of trials until the kitty reaches $\$n$ for the first time,
Y = number of trials until a jackpot of at least $\$n$ is won for the first time,
Z = number of trials until a jackpot of exactly $\$n$ is won for the first time.

Our intuition tells us that $E(X) < E(Y) < E(Z)$. More precisely, we will achieve our goals by showing

$$E(X) = (1 - q^n)/q^n p, \qquad E(Y) = 1/q^n p, \qquad E(Z) = 1/q^n p^2,$$

where $q = 1 - p$. To compute each of these three expectations, we will condition on the random variable W = number of trials until the first success occurs. We know that W has a geometric distribution with parameter p. Since the process restarts with an empty kitty following the first success, we find

$$E(X|W=w)=\begin{cases} w+E(X) & \text{for } w=1,2,\ldots,n \\ n & \text{for } w=n+1,n+2,\ldots \end{cases}$$

Therefore,

$$\begin{aligned} E(X) &= \sum_{w=1}^{\infty} E(X|W=w)P(W=w) \\ &= \sum_{w=1}^{n} (w+E(X))P(W=w)+nP(W\geqslant n+1) \\ &= \sum_{w=1}^{n} wq^{w-1}p+E(X)P(W\leqslant n)+nP(W\geqslant n+1). \end{aligned}$$

Solving for $E(X)$, we see that

$$E(X)=\frac{p\sum_{w=1}^{n} wq^{w-1}}{P(W\geqslant n+1)}+n=\frac{p\{(nq^{n+1}-(n+1)q^n+1)/p^2\}}{q^n}+n=\frac{1-q^n}{q^n p},$$

where the formula $\sum_{w=1}^{n} wq^{w-1}=(nq^{n+1}-(n+1)q^n+1)/p^2$ is obtained by differentiating both sides of the equation

$$1+q+q^2+\cdots+q^n=(1-q^{n+1})/(1-q)$$

with respect to q. Similarly,

$$\begin{aligned} E(Y) &= \sum_{w=1}^{\infty} E(Y|W=w)P(W=w) \\ &= \sum_{w=1}^{n} (w+E(Y))P(W=w)+\sum_{w=n+1}^{\infty} wP(W=w) \\ &= E(W)+E(Y)P(W\leqslant n), \end{aligned}$$

so we find

$$E(Y)=\frac{E(W)}{P(W\geqslant n+1)}=\frac{1/p}{q^n}=\frac{1}{q^n p}.$$

Finally,

$$\begin{aligned} E(Z) &= \sum_{w=1}^{\infty} E(Z|W=w)P(W=w) \\ &= \sum_{w\neq n+1} (w+E(Z))P(W=w)+(n+1)P(W=n+1) \\ &= E(W)+E(Z)P(W\neq n+1), \end{aligned}$$

from which we obtain

$$E(Z)=\frac{E(W)}{P(W=n+1)}=\frac{1/p}{q^n p}=\frac{1}{q^n p^2}.$$

Also solved by J. Michael Bossert, Richard Bradford (student), Amy Lindeman (student), Larry Lucas (part a), James M. Meehan (part a), William A. Newcomb, Jan Söderkvist (Sweden), Eric Wepsic (student), Elliott A. Weinstein, Western Maryland College Problems Group, and the proposer. There was one incomplete solution.

Convergence of products **June 1987**

1269. *Proposed by L. Matthew Christophe, Jr., Wilmington, Delaware.*

For each nonnegative real number x, let

$$a_n(x) = \frac{\prod_{k=1}^{n-1}(k+x)\prod_{k=2}^{n}(k+x)}{(n!)^2}$$

for $n = 1, 2, 3, \ldots$. Evaluate $\lim_{n\to\infty} a_n(x)$ as a function of x.

I. *Solution by Irl C. Bivens and Benjamin G. Klein, Davidson College, North Carolina.*

We will show that

$$\lim_{n\to\infty} a_n(x) = \begin{cases} 0 & 0 \leqslant x < 1/2, \\ \dfrac{8}{3\pi} & x = 1/2, \\ \infty & x > 1/2. \end{cases}$$

Consider $x = 1/2$. We see that

$$a_n(1/2) = \frac{(3\cdot 5\cdots(2n-1))(5\cdot 7\cdots(2n+1))}{(n!)^2 2^{2n-2}}$$

$$= \frac{\frac{4(2n+1)}{3}((2n)!)^2}{(n!)^4 2^{4n}}$$

$$= \frac{4(2n+1)}{3}\left(\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right)^2.$$

It follows readily from Stirling's formula that $\binom{2n}{n}(1/2)^{2n}$ is asymptotic to $(\pi n)^{-1/2}$, so that

$$\lim_{n\to\infty} a_n(1/2) = \lim_{n\to\infty}\frac{4(2n+1)}{3\pi n} = \frac{8}{3\pi},$$

as claimed.

Next, note that

$$a_n(x) = \prod_{k=2}^{n}\frac{(k-1+x)(k+x)}{k\cdot k}$$

$$= \prod_{k=2}^{n}\left(1+\frac{x-1}{k}\right)\left(1+\frac{x}{k}\right)$$

$$= \prod_{k=2}^{n}\left(1+\frac{(2x-1)k+x(x-1)}{k^2}\right). \qquad (*)$$

Since, for $x \neq 1/2$, the series

$$\sum_{k=2}^{\infty}\left(\frac{(2x-1)k+x(x-1)}{k^2}\right)$$

is divergent, the infinite product ($*$) above converges to 0 for $x < 1/2$ and diverges to ∞ for $x > 1/2$, as claimed. (Cf. Theorems 12-52 and 12-55 on pp. 381–2 of Apostol, *Mathematical Analysis*, Addison Wesley, 1957.)

II. *Solution by The SUNY II_1, Mathematics, Stony Brook, New York.*

We can rewrite a_n as

$$a_n(x) = \frac{(1+x)^2 \cdots (n+x)^2}{(1+x)(n+x)(n!)^2}.$$

On the other hand, by Euler's limit formula,

$$z\Gamma_n(z) = \frac{n!n^z}{(z+1)\cdots(z+n)}$$

converges to $z\Gamma(z)$ for $z \neq -1, -2, \ldots$. Hence, if z is a nonnegative real number, we have

$$a_n(z)z^2\Gamma_n^2(z) = \frac{n^{2z}}{(1+z)(n+z)} \to \begin{cases} 0 & 0 \leqslant z < 1/2, \\ \frac{2}{3} & z = 1/2, \\ \infty & z > 1/2. \end{cases}$$

Since $\Gamma(1/2) = \sqrt{\pi}$, we get the result given in the preceding solution.

Also solved by Miguel A. Arcones, Paul Bracken, Chico Problem Group, James Grochocinski, Ole Jørsboe (Denmark), Juan Manuel Guevara Jordan, Václav Konečný, L. Kuipers (Switzerland), Kee-Wai Lau (Hong Kong), William A. Newcomb, Volkhard Schindler (East Germany), H.-J. Seiffert (West Germany), Jan Söderkvist (Sweden), Michael Vowe (Switzerland), Elliott A. Weinstein, and the proposer. There was one incorrect solution.

The *Chico Problem Group* replaced x with a complex variable $z = x + iy$, and showed that the limit is 0 if $\mathrm{Re}(z) < 1/2$, $8/(3\pi)$ if $z = 1/2$, $+\infty$ if $z = x > 1/2$, and does not exist for other complex z.

Exponents in factorials **June 1987**

1270. *Proposed by Roger B. Eggleton, The University of Newcastle, Australia.*

Let $n \geqslant 1$ and $a \geqslant 2$ be integers. For which values of a and n is $(n+1)!$ a multiple of a^n?

Solution by Adam Riese, Wright State University, Dayton, Ohio.

For a prime p, let k_p be the exponent of p in the prime factorization of $(n+1)!$; that is, p^{k_p} divides $(n+1)!$ but p^{k_p+1} does not.

Let t be an integer such that $p^t \leqslant n+1 < p^{t+1}$. Then

$$k_p = \sum_{j=1}^{t} \left\lfloor \frac{n+1}{p^j} \right\rfloor \leqslant \sum_{j=1}^{t} \frac{n+1}{p^j}$$

$$= (n+1)\frac{1}{p}\frac{1-p^{-t}}{1-p^{-1}} \leqslant (n+1)\frac{1}{p}\frac{1-\dfrac{1}{n+1}}{1-\dfrac{1}{p}} = \frac{n}{p-1} \leqslant n.$$

If $p \geqslant 3$ then $k_p < n$, and if $p = 2$ then $k_p = n$ if and only if $n + 1 = p^t$.

Also solved by Anders Bager (Denmark), Seung Jin Bang (Korea), S. F. Barger, Merrill Barnebey, Andreas Bender (student, Switzerland), Chico Problem Group, Hugh M. Edgar, Alberto Facchini (Italy), *James Grochocinski, Thomas Hofmeister (student, FR Germany), Václav Konečný, L. Kuipers (Switzerland), Y. H. Harris Kwong, Kee-Wai Lau (Hong Kong), Woei Lin and Rebecca Lee, Helen M. Marston, John McCleary, Middle Tennessee State University Problem Solving Group, William A. Newcomb, Amir Akbary Majdabad No (student, Iran), Florentin Smarandache (Romania), Jan Söderkvist (Sweden), Edward T. H. Wang (Canada), Eric Wepsic (student), C. Wildhagen (The Netherlands), and the proposer.* There were three incomplete solutions.

Smarandache considered the general problem of finding positive integers n, a, and k, so that $(n+k)!$ should be a multiple of a^n. Also, for positive integers p and k, with p prime, he found a formula for determining the smallest integer $f(k)$ with the property that $(f(k))!$ is a multiple of p^k.

A bouncing ball problem **June 1987**

1271. *Proposed by Skip Thompson, Oak Ridge National Laboratory, Oak Ridge, Tennessee.*

A ball is released from rest at position (x_0, y_0), where $0 \leqslant x_0 < 1$ and $x_0 + y_0 > 1$. Its acceleration due to gravity is the constant vector $(0, -g)$. The ball repeatedly bounces on the ramp $x + y = 1$ $(0 \leqslant x \leqslant 1)$. Suppose that every bounce has the effect of "switching" the velocity vector from (u, v) just before the collision to $(-kv, -ku)$ just after the collision, where k is a constant between 0 and 1. (The case of $k = 1$ corresponds to a perfectly elastic collision.) Determine all values of (x_0, y_0) for which the x-coordinate of the ball will never exceed 1, i.e., the ball will not go beyond the bottom of the ramp.

Solution by the proposer.

The initial trajectory is

$$x = x_0, \qquad y = y_0 - gt^2/2, \tag{1}$$

so the ball hits the ramp $x + y = 1$ for the first time at $t = T$, where

$$T = \left(2(x_0 + y_0 - 1)/g\right)^{1/2}. \tag{2}$$

Now suppose that the ball is initially at position $(a, 1-a)$ on the ramp and its velocity vector is (v_x, v_y). Then it will follow the trajectory

$$x = a + v_x t, \qquad y = 1 - a + v_y t - gt^2/2,$$

and again hit the ramp $x + y = 1$ after an elapsed time

$$\tau = 2(v_x + v_y)/g. \tag{3}$$

Starting from $t = 0$ when the ball is released, let t_n denote the time at which the nth bounce occurs and let us refer to the interval (t_n, t_{n+1}) as the nth *epoch*. From (1) and (2), we know that at the beginning of the first epoch, $v_x = -k(-gT)$ and $v_y = 0$. We claim that, in general, the velocity components at the beginning of the nth epoch are

$$v_x = ngTk^n, \qquad v_y = -(n-1)gTk^n. \tag{4}$$

The proof is an easy induction. Assuming (4) for $n = i$, we see that it follows from (3) that the length of the ith epoch is

$$t_{i+1} - t_i = 2k^iT. \tag{5}$$

Within the ith epoch, v_x remains constant $igTk^i$, while v_y decreases by $g(t_{i+1} - t_i) = 2gTk^i$. Thus, at the end of the ith epoch, $v_y = -(i+1)gTk^i$. Now the switching condition at impact yields the fact that (4) holds for $n=i+1$, completing the proof.

Since v_x is constant within an epoch, we have the fact that the x-coordinate of the ball at the end of the nth epoch is

$$x_n = x_0 + \sum_{i=1}^{n} (igTk^i)(2k^iT),$$

as long as $x_n \leqslant 1$. Indeed

$$\lim_{n\to\infty} x_n = x_0 + \frac{2gTk^2}{(1-k^2)^2} = x_0 + \frac{4(x_0+y_0-1)k^2}{(1-k^2)^2} \leqslant 1$$

if and only if

$$y_0 \leqslant (1-x_0)\left(\frac{k^2+1}{2k}\right)^2.$$

Also solved by Seung Jin Bang (Korea), J. C. Binz (Switzerland, two solutions), Milton P. Eisner, Václav Konečný, L. Kuipers (Switzerland) and M. Kuipers (The Netherlands), Willard L. Maier, Peter N. Meisinger, Volkhard Schindler (East Germany), William A. Newcomb, Stephen Noltie, Jan Söderkvist (Sweden), George Vakanas (student), Western Maryland College Problems Group, and the proposer (two solutions).

Answers

Solutions to the Quickies on *p. 196.*

A734. Geometrically, if one wants the locus of points which are the orthogonal projections of the fixed point (h,k,l) on all lines through the origin with direction numbers (ξ,η,ζ), one can obtain the given parametric representation. From this interpretation, it follows quickly that the locus is that portion of the sphere with the segment joining the origin to (h,k,l) as a diameter and which lies in the first orthant.

Alternatively, it follows that

$$x^2+y^2+z^2 = hx+ky+lz.$$

Thus the surface is that part of the sphere with radius $r=\sqrt{h^2+k^2+l^2}/2$ and center $(h/2, k/2, l/2)$, which lies in the first orthant.

A735.

$$\begin{aligned} &a+b+c-3(abc)^{1/3} \\ &\quad = (a^{1/3}+b^{1/3}+c^{1/3})(a^{2/3}+b^{2/3}+c^{2/3}-a^{1/3}b^{1/3}-a^{1/3}c^{1/3}-b^{1/3}c^{1/3}). \end{aligned}$$

Since

$$a^{2/3}+b^{2/3}+c^{2/3} \geqslant a^{1/3}b^{1/3}+a^{1/3}c^{1/3}+b^{1/3}c^{1/3}$$

with equality if and only if $a=b=c$, it follows that

$$a^{2/3}+b^{2/3}+c^{2/3}-a^{1/3}b^{1/3}-a^{1/3}c^{1/3}-b^{1/3}c^{1/3}$$

is always positive and hence $a+b+c-3(abc)^{1/3}=0$ if and only if

$$a^{1/3}+b^{1/3}+c^{1/3}=0.$$

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Dudley, Underwood, *A Budget of Trisections*, Springer-Verlag, 1987; xv + 169 pp, $29.80.

"What follows ... is an effort to do something which may be as impossible as trisecting the angle: namely to put an end to trisections and trisectors"—as De Morgan's Budget of Paradoxes may have done to circle squaring and squarers a century ago. Dudley shows how to trisect the angle (using noneuclidean tools), cites characteristics of trisectors and advises how to handle them, gives anecdotes of personal encounters with three, and gives succinct summaries of many trisection attempts (including one by "U.D."). He also makes us all an appeal: "If, reader, you know of any mathematical crank literature, I would be pleased to have it, or a copy of it: I have the collector's lust. I want it *all*." I read this book JUST IN TIME--I fished out of my wastebasket a crank publication that he would have missed forever if not for my reading that appeal this very night. But Woody, why won't you tell the reason why cranks grip you so? And after the trisectors go away, watch out for the guys with the one-page proofs of the Four Color Theorem ...

Gleick, James, *Chaos: Making a New Science*, Viking, 1987; xi + 352 pp, $19.95.

Astonishingly engrossing book about the discoverers and discovery of chaos, with its concomitant phenomena of fractals and strange attractors. Chaos emerges as an interdisciplinary scientific endeavor, a significant cross-pollinizing influence. Gleick, science writer for the New York Times, has followed the Martin Gardner tradition of writing about mathematics without using equations. With luck, this book will be the best-read popular science book of the year.

Tymoczko, Thomas (ed.), *New Directions in the Philosophy of Mathematics*, Birkhäuser, 1986; xvii + 323 pp, $57.50.

The "new" of the title of this handy anthology must be interpreted in the timeframe of philosophy, as all but two of the essays date to 1979 or earlier and none of them is original with this volume. The book represents two parts of a larger three-part project. Included here are essays challenging "the dogmas underlying foundationalist views of mathematics," and others focussing on mathematics as it is practiced; a final part was to have reflected on the impact of recent discoveries in mathematical logic. Tymoczko has spliced in introductions, and the endnotes for each essay are apparently also due to him (a circumstance that should be made clear to the reader, so as not to mistake their authorship).

Stewart, Ian, *The Problems of Mathematics*, Oxford Univ Pr, 1987; ix + 257 pp, $32.50, $11.95(P).

A celebration of life in the Golden Age of Mathematics--now!--in vivid language, by one of the best expositors of mathematics. The cover calls this book "an astonishingly racy account" of some of the central problems and achievements of modern mathematics. This is a marvelous book to inspire both mathematics majors and nonmajors (not to mention your mathematically illiterate colleagues in other fields). With a racier title ("Modern Marvels of Mathematics" ?), a few more illustrations, a more exciting cover, and a price of $3.95 in paper, this book would have a fighting chance in drugstores, supermarkets, and bookshops everywhere. Meanwhile, it is great reading for courses in liberal arts mathematics.

Hilts, Philip J., Age-old math problem finally may have solution, *Capital Times* (Madison, WI) (9 March 1988) 1, 16.
Gleick, James, Mathematics expert may soon resolve a 350-year problem, *New York Times* (10 March 1988) A18.
Peterson, I., Fermat's last theorem: A promising approach, *Science News* 133 (19 March 1988) 180-181.
Solving the puzzle, *Time* (21 March 1988) 64.
Gleick, James, Fermat's theorem solved? Not this time, experts say, *New York Times* (29 March 1988) 23.
Peterson, I., Doubts about Fermat solution, *Science News* 133 (8 April 1988) 230.
Krauthammer, Charles, The joy of math, or Fermat's revenge, *Time* (18 April 1988) 92.
Gleick, James, Fermat's last theorem still has 0 solutions, *New York Times* (17 April 1988) E28.

Reading the sequence of headlines tells the story in a nutshell; as Krauthammer puts it, "For one brief shining moment, it appeared as if the 20th century had justified itself." But the proof of Fermat's last theorem that was advanced by Yoichi Miyaoka (Tokyo Metropolitan Univ.), a specialist in algebraic geometry, has a serious gap. Krauthammer's reflections, however, are a paean to the sublimity of the art of mathematics.

Cipra, Barry, Zeroing in on the zeta function, *Science* 239 (11 March 1988) 1241-1242.

Survey of the meaning and significance of the Riemann hypothesis, citing recent progress. Current status: The first 1.5 billion zeros of the zeta function lie on the critical line, and new methods of A. Odlyzko (Bell Labs) and A. Schoenhage (Tübingen) allow examination of data on zeros as far out as the 10^{18}th.

Peterson, I., Math society says no to SDI funding, *Science News* 133 (2 April 1988) 213.

A referendum among members of the American Mathematical Society will keep the organization from making efforts to secure Star Wars funding for its members. The resolution was approved by a 57% majority in a mail ballot. A second resolution calling for greater effort to decrease the proportion of mathematics research funding coming from the U.S. Dept. of Defense (currently 40%) was approved by 74% of those voting. Approximately one-third of AMS members voted; voting was open to all members of the AMS, including foreign nationals and members residing outside the U.S.

Monastyrsky, Michael, *Riemann, Topology, and Physics*, Birkhäuser, 1987; xiii + 158 pp, $39.50.

"Soviet citizens can buy Monastyrsky's biography of Riemann for eleven kopecks. This translated edition will cost considerably more, but it is still good value for the money." So begins the introduction by Freeman Dyson. In fact, the book also contains, "in the bargain," Monastyrsky's monograph on topological themes in physics, a splendidly readable account.

Halmos, Paul R., *I Have a Photographic Memory*, AMS, 1987; iv + 326 pp, $58.

More than 600 photographs of mathematicians, taken over four decades by Paul Halmos. If you're not in this book, somebody you know is.

Johnson, George, Goldbach's conjecture: This one may be provable but we may never know, *New York Times* (17 April 1988) E28.

Notes early 20th-century result that every even integer is the sum of no more than 800,000 primes. Goes on to suggest that Gödel's theorem may mean we will never know if Goldbach's conjecture is true. The author is unaware, however, of the highly-encouraging result of Ching-jun Chen (1966) that every sufficiently large, even integer is the sum of a prime and an integer with at most two prime factors.

Beltrami, Edward, *Mathematics for Dynamic Modeling*, Academic, 1987; xvi + 277 pp, $27.50

Text for upper undergraduate and first-year graduate courses in modeling, or for an applications-oriented second course in differential equations--the book is definitely oriented toward differential equations. The topics are contemporary: equilibrium and stability, bifurcation, limit cycles, chaos, shock phenomena, catastrophe theory, and strange attractors. Applications are mainly to the physical sciences but also include interacting populations, tidal dynamics, traffic waves, river pollution, the heart as pump, and fish harvesting.

Kohl, Herbert, *Mathematical Puzzlements: Play and Invention in the Mathematical Field*, Schocken, 1987; x + 225 pp, $18.95.

Delightful collection of popular mathematics, by the author of *The Open Classroom*. Topics are patterns on the plane; number patterns; knots, maps and connections; and logic and strategy. Kohl credits one of his high-school math teachers for the inspiration to write this book.

Eves, Howard W., *Return to Mathematical Circles: A Fifth Collection of Mathematical Stories and Anecdotes*, PWS-Kent, 1988; xxi + 181 pp.

Another welcome collection of mathematical vignettes and trivia, many centering on Einstein.

Gindikin, Semyon Grigorevich, *Tales of Physicists and Mathematicians*, Birkhäuser, 1988; xi + 157 pp, $29.50.

Originally written for students in the Soviet Union, this book presents lively accounts of the lives and achievements of Cardano, Tartaglia, Galileo, Huygens, Pascal, and Gauss.

Gjertsen, Derek, *The Newton Handbook*, Methuen, 1986; xiv + 665 pp, $59.95.

Contains summaries of Newton's works amid 500 dictionary-like entries that cover the whole of what we know of Newton: his friends and acquaintances, his ideas, his artistic interests, his breakdown, his finances, and his funeral. If he had had a dog, its name would be here.

Peterson, I., Following pi down the decimal trail, *Science News* 133 (2 April 1988) 215.

Y. Kanada (Tokyo) has calculated pi to 200 million places and plans to reach 400 million by year's end. Why? "[B]ecause it's there," he says.

Berger, James O., and Donald A. Berry, Statistical analysis and the illusion of objectivity, *American Scientist* 76 (March-April 1988) 159-165.

Criticizes "standard statistics" for false pretensions to objectivity, and advocates Bayesian methods instead.

Peterson, I., Priming for a lucky strike, *Science News* 133 (6 February 1988) 85.

W. N. Colquitt (Houston Area Research Center) and L. Welsh, Jr., have found the 31st Mersenne prime:

$$2^{110503} - 1$$

This one occurs in a gap between two previously-known Mersenne primes, making it the third largest known.

Tomorrow's shapes: The practical fractal, *The Economist* (26 December 1987) 99-103.

Cites new discoveries in science that have resulted from considering fractal models: protein surfaces, battery corrosion, viscous fingering of oil, epidemics, distribution of galaxies, rainfall maps, noise. "The bravest hope for fractals is that they will unravel the mysteries of chaos."

NEWS & LETTERS

LETTER TO THE EDITOR

Dear Editor:

Here is an easy direct proof of the assertion that n divides the binomial coefficient $\binom{2n-2}{n-1}$. On page 306 of the December 1987 issue it is asserted that a simple direct proof may not exist.

Let p be any prime that divides n α times but not $\alpha + 1$ times. $(p^{\alpha} \| n)$. The assertion $n \mid \binom{2n-2}{n-1}$ is the same as $[n(n-1)!(n-1)!] \mid (2n-2)!$. So it has to be proved that

$$\sum_{m=1}^{\infty} \left[\frac{2n-2}{p^m}\right] \geq \alpha + 2 \sum_{m=1}^{\infty} \left[\frac{n-1}{p^m}\right].$$

It suffices to note that for each m, $1 \leq m \leq \alpha$, the relation $\left[\frac{2n-2}{p^m}\right] = 1 + 2\left[\frac{n-1}{p^m}\right]$ holds. This relation is almost obvious, since $n = p^{\alpha} t$ by hypothesis:

$$\left[\frac{2n-2}{p^m}\right] = \left[2p^{\alpha-m}t - 2/p^m\right] = 2p^{\alpha-m}t - 1;$$

$$\left[\frac{n-1}{p^m}\right] = p^{\alpha-m} - 1. \square$$

J.L. Brenner
R.J. Evans

REPLY FROM AUTHORS

This proof depends on the fact that the power of p dividing $k!$ is exactly $\sum_{m=1}^{\infty} \left[\frac{k}{p^m}\right]$. This is far deeper than the computation of the (ordinary) Catalan numbers. We are happy to have this confirmation of our belief that "no simple arithmetical proof seems available" of the fact that n divides $\binom{2n-2}{n-1}$.

Peter Hilton
Jean J. Pedersen

G.H. Hardy did not particularly esteem the Ph.D. degree or take it very seriously, and apparently was not above writing a thesis for a student. I am told that he once did this for an Indian student, who then asked Hardy to write a letter saying that he had written a good thesis (so that he could get a better job). Hardy balked at this, but told the student to show the thesis to Littlewood and make the same request. Littlewood wrote the letter and the student got the job. I told this story to an Indian friend, who at once said, "Oh, I know that man." But he wouldn't tell me who it was.

—R.P. Boas

During my last year of graduate work, I gave a lecture to the Harvard Colloquium. Afterwards, Marshall Stone asked me what I was going to do next. I told him, adding some diffident words about not supposing that the problem was very interesting. He snapped, "If it isn't interesting, why do it?" I took that advice to heart, and thereafter never said "It may be interesting that..." but, firmly, "It *is* interesting that."

It is not often that one can think of an apt reply to a comment, so I am rather proud of one time that I could. Somebody came up to me after a talk I had given, and said, "You make mathematics seem like fun." I was inspired to reply, "If it isn't fun, why do it?"

—R.P. Boas

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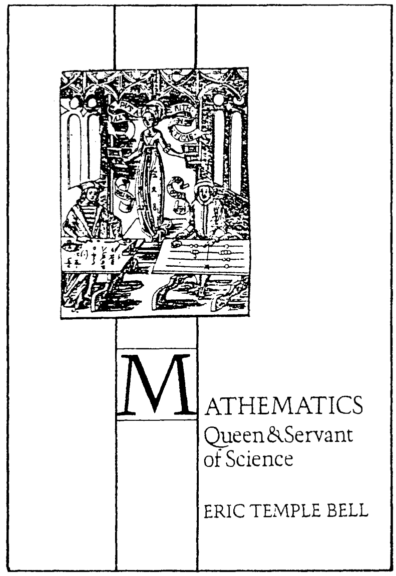
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